SUMMATION COMPARISON THEOREMS
FOR HALF-LINEAR
SECOND ORDER DIFFERENCE EQUATIONS
ON FINITE INTERVAL

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Abstract

MAŘÍK, R.:*Summation comparison theorems for half-linear second order difference equations on finite interval.*

In the paper, new comparison theorems for the half-linear difference equation

$$\Delta\Big(R_k\Phi(\Delta z_k)\Big)+C_k\Phi(z_{k+1})=0,\quad \Phi(u)=|u|^{p-2}u,\quad p>1,$$

are derived. We show that if a solution of this equation has a generalized zero on the discrete interval $[a,b]$, then the same holds for a solution of its majorant. The main tool used in the paper is the variational technique which relates nonexistence of a solution with a generalized zero with nonegativity of the p-degree functional defined on the suitable class of admissible functions.

difference equation, second order, focal point, half-linear equation, p-degree functional, free end point

MSC2000: 34A10

1. INTRODUCTION

Consider the second order half-linear difference equation

$$\Delta\Big(R_k\Phi(\Delta x_k)\Big)+C_k\Phi(x_{k+1})=0, \tag{1}$$

where Δ is the forward difference operator, $\{C_k\}$, $\{R_k\}$ are real sequences, $R_k\neq 0$ for $k=0,\dots,n+1$, and $\Phi(u)=|u|^{p-2}u$, $p>1$, is a power type nonlinearity. The study of equation (1) has been initiated in Řehák (2001) and the most important results are summarized in Chapter 8 of the monograph Došlý, Řehák (2005).

Despite the lack of linearity, a constant multiple of any solution of (1) is also a solution and equation (1) has one half of linearity properties. It is well known that there is a close similarity between equation (1) and the linear second order difference equation. In particular, many results from oscillation theory of second order linear difference equations can be extended to (1). These oscillation and

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nonoscillation results are frequently based on a comparison of two equations on the infinite interval. The aim of this paper is to derive comparison theorems which compare (1) with another half-linear difference equation

$$\Delta\Big(r_k\Phi(\Delta y_k)\Big)+c_k\Phi(y_{k+1})=0 \tag{2}$$

on a finite interval.

First, let us recall the definition of a generalized zero, which is (from the point of view of the Sturmian comparison theory) a natural replacement for zeros of solutions to differential equations. Remark that, unless stated explicitly otherwise, under the interval $[m,n]$ we actually mean the discrete set $\{m,m+1,\dots n\}$. In a similar way we work also with other intervals.

Definition 1. The interval $(m,m+1]$ is said to contain a *generalized zero* of a solution $x=\{x_k\}$ of Eq. (1) if $x_m\neq 0$ and $R_mx_mx_{m+1}\leq 0$.

It is well known, see for example Řehák (2001) or Došlý, Řehák (2005), that equation (1) tends to have more generalized zeros than (2), if the inequalities $R_i\leq r_i$ and $C_i\geq c_i$ are satisfied. In contrast to the pointwise comparison we formulate our results more generally in terms of sums of the coefficients C_i and c_i. Our aim is to derive a discrete version of the following theorem due to Leighton.

Theorem A (Leighton (1983), Theorem 1.1)**.** *Let $p(t)$ and $q(t)$ be piecewise continuous on $[a,f]$ with $q(x)\geq 0$ there, and suppose that*

$$\int_a^x q(t)\,\mathrm{d}t\leq\int_a^x p(t)\,\mathrm{d}t,\quad a\leq x\leq f;\quad p(x)\not\equiv q(x)$$

holds. If equation

$$u''+q(t)u=0$$

has a solution $u(t)$ with the property that $u'(a)=u(f)=0$, $u(x)>0$ on the interval of real numbers $[a,f)$, a solution $v(t)$ of

$$v''+p(t)v=0$$

with $v(a)>0$, $v'(a)\leq 0$ must have a zero on the interval of real numbers (a,f).

Another aspect which makes our results different from those published in the literature is that similarly as in Theorem A we compare two solutions which do not vanish at the left end point of the interval. As far as the author knows, the results are new even for the linear diference equation.

The main tool used in the paper is the variational technique which relates equation (1) and the corresponding discrete scalar p-degree functional

$$J(\eta)\equiv A|\eta_0|^p+\sum_{k=0}^{n}\Big(R_k|\Delta\eta_k|^p-C_k|\eta_{k+1}|^p\Big),\qquad A\in\mathbb{R} \tag{3}$$

defined on the class of nontrivial sequences $\{\eta_k\}_{k=0}^{n+1}$ such that $\eta_{n+1}=0$. Note that since we aim to compare the solutions which do not vanish at the left end point of the interval, we drop the usual requirement $\eta_0=0$ from the definition of the admissible sequences for functional J and also include the term $A|\eta_0|^p$. The relationship between the half-linear difference equation and the p-degree functional is established in the following theorem.

Theorem B (Mařík (2000), Theorem 1)**.** *The following statements are equivalent:*

(i) *The solution* $x = \{x_k\}$ *of Eq.* (1) *given by* $R_0\Phi\Big(\frac{\Delta x_0}{x_0}\Big) = A$ *has no generalized zero on* $(0, n+1]$.

(ii) *Functional* (3) *is positive definite on the class of nontrivial sequences* $\eta = \{\eta_k\}_{k=0}^{n+1}$ *satisfying* $\eta_{n+1} = 0$

The following result allows to compare two solutions of two different equations and it is an immediate consequence of Theorem B. The crucial aspect of the proof of this theorem lies in the fact that the functional J vanishes for the sequence which solves equation (1) and satisfies initial condition closely connected with the value a.

Theorem C (Leighton type comparison theorem, Mařík (2000), Corollary 1)**.** *Let* $y = \{y_k\}$ *be a solution of Eq.* (2), *such that* $y_{n+1} = 0 \neq y_0$ *and* $a := r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. *Let* A *be such that*

$$V(y) := (A-a)|y_0|^p + \sum_{k=0}^{n}\Big[(R_k - r_k)|\Delta y_k|^p - (C_k - c_k)|y_{k+1}|^p\Big] \leq 0.$$

Then the solution $x = \{x_k\}$ *of Eq.* (1) *given by* $R_0\Phi\big(\frac{\Delta x_0}{x_0}\big) = A$ *has a generalized zero on* $(0, n+1]$, *i.e., there exists* $i \in (0, n]$ *such that* $x_i \neq 0$ *and* $R_i x_i x_{i+1} \leq 0$ *holds.*

2. MAIN RESULTS

This section contains the main results of the paper. In the following theorem we prove that if the solution of half-linear differential equation (2) vanishes at the point $n+1$, then the solution of the equation (1) with a sufficiently large coefficient C_i has a generalized zero on $(0, n+1]$, if the initial difference is negative and not too large. However, the words "sufficiently large" are here in the integral sense as (4) shows. Hence the inequality $C_i \geq c_i$ is not necessary for all i.

Note the technical assumption on nonegativity of c_i which assures that the solution which starts with positive initial value and nonpositive initial difference is nonincreasing.

Theorem 1. *Let* $r_k > 0$ *on* $[0, n]$, $c_k \geq 0$ *on* $[0, n-1]$, $c_0 > 0$. *Let* y *be a solution of* (2) *on* $[0, n-1]$ *such that* $y_0 \geq y_1 > 0$, $y_k > 0$ *on* $[0, n]$ *and* $y_{n+1} = 0$. *Denote* $a = r_0\Phi\left(\frac{\Delta y_0}{y_0}\right)$. *Let* $R_k \leq r_k$ *on* $[0, n]$, $C_0 \geq c_0$, $A \leq a$ *and*

$$\left|\frac{y_0}{y_1}\right|^p \Big(a - A\Big) + \sum_{i=0}^{k}\Big(C_i - c_i\Big) \geq 0 \tag{4}$$

for $k \in [0, n-1]$. *Then the solution* $z = \{z_k\}$ *of* (1) *given by the conditions* $z_0 > 0$, $A = R_0\Phi\left(\frac{\Delta z_0}{z_0}\right)$ *has a generalized zero on* $(0, n+1]$, *i.e., there exists* $i \in [0, n]$ *such that* $z_i \neq 0$ *and* $R_i z_i z_{i+1} \leq 0$.

Proof. From $R_k \leq r_k$ we get

$$V(y) \leq (A-a)|y_0|^p + \sum_{k=0}^{n}(c_k - C_k)|y_{k+1}|^p.$$

Further, from (2) it follows

$$\Phi(\Delta y_{k+1}) = \frac{r_k}{r_{k+1}}\Phi(\Delta y_k) - \frac{c_k}{r_{k+1}}\Phi(y_{k+1})$$

for $k \in [0, n-1]$ and

$$\Phi(\Delta y_{k+1}) < 0$$

for $k \in [0, n-1]$. Hence $|y_{k+1}|^p$ is decreasing on $[0, n]$. Clearly there exists $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, such that the intervals of real numbers $I_k := (|y_k|^p - \varepsilon, |y_k|^p + \varepsilon) \subseteq \mathbb{R}^+$, $k \in [1, n]$, satisfy $I_j \cap I_k = \emptyset$ for $j \neq k$. In each I_k let us choose α_k, $\alpha_k \in I_k \cap \mathbb{Q}^+$, such that

$$(c_k - C_k)|y_{k+1}|^p \le (c_k - C_k)\alpha_{k+1} \qquad \text{for } k \in [0, n-1]. \tag{5}$$

Denote by β the least common multiple of denominators of α_k. Then the numbers β_k defined by $\beta_k = \beta\alpha_k$ form a decreasing sequence for $k \in [1, n]$ and $\beta_k \in \mathbb{N}$. Combining these computations with $y_{n+1} = 0$ we obtain

$$\begin{aligned} V(y) &\le (A - a)|y_0|^p + \sum_{k=0}^{n-1}(c_k - C_k)|y_{k+1}|^p \\ &\le (A - a)|y_0|^p + \sum_{k=0}^{n-1}(c_k - C_k)\alpha_{k+1} \\ &\le (A - a)|y_0|^p + \frac{1}{\beta}\sum_{k=0}^{n-1}(c_k - C_k)\beta_{k+1} \\ &= (A - a)|y_0|^p + \frac{1}{\beta}\sum_{k=0}^{n-1}\sum_{i=1}^{\beta_{k+1}}(c_k - C_k) \end{aligned}$$

Changing the order of summation we get

$$V(y) \le (A - a)|y_0|^p + \frac{1}{\beta}\sum_{i=1}^{\beta_1}\sum_{k=0}^{\gamma_i}(c_k - C_k),$$

where γ_i is a well defined number from the discrete interval $[0, n-1]$. More precisely, γ_k denotes how many times the number k appears in the double sum $\sum_{k=0}^{n-1}\sum_{i=1}^{\beta_{k+1}} k$. By (4), we obtain

$$\begin{aligned} V(y) &\le (A - a)|y_0|^p + \frac{1}{\beta}\sum_{i=1}^{\beta_1}\left|\frac{y_0}{y_1}\right|^p(a - A) \\ &= (A - a)|y_0|^p + \alpha_1\left|\frac{y_0}{y_1}\right|^p(a - A) \\ &= |y_0|^p(a - A)\frac{\alpha_1 - |y_1|^p}{|y_1|^p}. \end{aligned}$$

Since (5) and $C_0 \ge c_0$ imply $\alpha_1 \le |y_1|^p$, we have $V(y) \le 0$. Now the statement follows from Theorem C. $\square$

There is a variant of Theorem 1 which is based on the nonnegativity of slightly different sum than (4). Namely, the coefficient c_k has the weight $\frac{R_{k+1}}{r_{k+1}}$ in this sum. To derive this modification of Theorem 1 we need the following Lemma 1. This lemma is proved in Mařík (2000), Corollary 3, as a corollary of Theorem B. However, the original version contains some misprints and for this reason we restate this lemma with a shorter proof than the proof presented in Mařík (2000).

Lemma 1. *Let* $y = \{y_k\}$ *be a solution of Eq.* (2) *on* $[0, n-1]$, *such that* $y_{n+1} = 0 \neq y_0$ *and* $a = r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. *Let* A *be such that*

$$\widetilde{V}(y) := \Big(A - \frac{R_0}{r_0}a\Big)|y_0|^p - \sum_{k=0}^{n-1}\Big\{\Delta\Big(\frac{R_k}{r_k}\Big)r_k\Phi(\Delta y_k)y_{k+1} + \Big(C_k - \frac{R_{k+1}}{r_{k+1}}c_k\Big)|y_{k+1}|^p\Big\} \le 0.$$

Then the solution $z = \{z_k\}$ *of Eq.* (1) *given by* $R_0\Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ *has a generalized zero on* $(0, n+1]$, *i.e., there exists* $i \in [0, n]$ *such that* $R_i z_i z_{i+1} \le 0$ *holds.*

Proof. Let $y = \{y_k\}$ be a solution of (2) on $[0, n-1]$ which satisfies $y_{n+1} = 0 \neq y_0$ and $a = r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. Then

$$\begin{aligned} L[y_k] &\equiv \Delta\big(R_k\Phi(\Delta y_k)\big) + C_k\Phi\big(y_{k+1}\big) = \Delta\Big(\frac{R_k}{r_k}r_k\Phi(\Delta y_k)\Big) + C_k\Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big)r_k\Phi(\Delta y_k) + \frac{R_{k+1}}{r_{k+1}}\Delta\Big(r_k\Phi(\Delta y_k)\Big) + C_k\Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big)r_k\Phi(\Delta y_k) + \Phi(y_{k+1})\Big[C_k - \frac{R_{k+1}}{r_{k+1}}c_k\Big] \end{aligned} \tag{6}$$

holds for $k \in [0, n-1]$. Using summation by parts we get

$$\begin{aligned} \sum_{k=0}^{n} y_{k+1}L[y_k] &= \sum_{k=0}^{n} y_{k+1}\Big\{\Delta(R_k\Phi(\Delta y_k)) + C_k|y_{k+1}|^p\Big\} \\ &= R_{n+1}\Phi(\Delta y_{n+1})y_{n+1} - R_0\Phi(\Delta y_0)y_0 - \sum_{k=0}^{n}\Big[R_k|\Delta y_k|^p - C_k|y_{k+1}|^p\Big] \end{aligned}$$

Therefore in view of (6) and $y_{n+1} = 0$, clearly

$$\begin{aligned} J(y) &= |y_0|^p\Big[A - R_0\Phi\Big(\frac{\Delta y_0}{y_0}\Big)\Big] - \sum_{n=0}^{n} y_{k+1}L[y_k] \\ &= \Big(A - \frac{R_0}{r_0}a\Big)|y_0|^p - \sum_{n=0}^{n-1} y_{k+1}L[y_k] \\ &= \Big(A - \frac{R_0}{r_0}a\Big)|y_0|^p - \sum_{k=0}^{n-1}\Big\{\Delta\Big(\frac{R_k}{r_k}\Big)r_k\Phi(\Delta y_k)y_{k+1} + \Big(C_k - \frac{R_{k+1}}{r_{k+1}}c_k\Big)|y_{k+1}|^p\Big\} \\ &\le 0. \end{aligned}$$

and the statement follows from Theorem B. □

Theorem 2. *Let* $r_k > 0$ *on* $[0, n]$, $c_k \ge 0$ *on* $[0, n-1]$, $C_0 > \frac{R_1}{r_1}c_0$, $\Delta\frac{R_k}{r_k} \le 0$. *Let* $y = \{y_k\}$ *be a solution of* (2) *on* $[0, n-1]$, *such that* $y_0 \ge y_1 > 0$, $y_k > 0$ *on* $[0, n]$, $y_{n+1} = 0$. *Suppose that* $A > \frac{R_0}{r_0}a$,

$$\left|\frac{y_0}{y_1}\right|^p\Big(\frac{R_0}{r_0}a - A\Big) + \sum_{i=0}^{k}\Big(C_i - \frac{R_{i+1}}{r_{i+1}}c_i\Big) \ge 0$$

for $k \in [0, n-1]$. *Then the solution* $x = \{x_k\}$ *of* (1) *given by the condition* $x_0 > 0$, $A = R_0\Phi\left(\frac{\Delta x_0}{x_0}\right)$ *has a generalized zero on* $[0, n+1]$, *i.e., there exists* $i \in [0, n]$ *such that* $x_i \neq 0$ *and* $R_i x_i x_{i+1} \le 0$.

Proof. From the assumption $\Delta \frac{R_k}{r_k} \leq 0$ we get

$$\widetilde{V}(y) \leq \Big(A - \frac{R_0}{r_0} a\Big)|y_0|^p - \sum_{k=0}^{n-1}\Big(C_k - \frac{R_{k+1}}{r_{k+1}} c_k\Big)|y_{k+1}|^p$$

The remaining part of the proof is essentially similar to the proof of Theorem 1 where we replace $V(y)$, a and c_k by $\widetilde{V}(y)$, $\frac{R_0}{r_0}a$ and $\frac{R_{k+1}}{r_{k+1}}c_k$, respectively. ☐

3. Summary

The classical results in the comparison theory of half-linear differential and difference equations deal with the generalized zeros of solutions which vanish at the left end point of the interval. Focal points, i.e. generalized zeros of solutions which start with zero difference, can be considered as a natural continuation of this research. The results presented in this paper include focal points if we choose $A = a = 0$ in Theorems 1 and 2.

Another companion of the conjugate point and the focal point is also the so called coupled point, the point associated with functional defined on another class of admissible functions, such as functional with free end points. Theory of discrete coupled points has been introduced in a series of papers by Hilscher and Zeidan, see Hilscher, Zeidan (2002, 2004, 2005) and the references therein. The possible extension of coupled point to half-linear equation and possibility to formulate comparison theorems in terms of coupled points is still an open question and a subject of the current research.

Further, there are results from the theory of differential equations, which allow to study nonoscillatory half-linear differential equations as a perturbation of another half-linear equation. This technique has been started in the paper Elbert, Schneider (2000) and extended in a series of papers by Došlý and coauthors. Among others, it has been shown that this method extends to difference, see e.g. Došlý and Fišnarová (2008, 2009), and can be formulated in variational setting, see Došlý and Fišnarová (2011). We hope that developing similar method for functional (3) instead of functional with zero end points opens the door to future extensions.

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