Half-linear ODE and modified Riccati equation: Comparison theorems, integral characterization of principal solution☆

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Abstract

In the paper we study the half-linear differential equation with one dimensional p-Laplacian

$$(r(t)\Phi_p(x'))' + c(t)\Phi_p(x) = 0,$$

where $\Phi_p(x) = |x|^{p-2}x$ and $p > 1$. Using a suitable modification of the so-called linearization technique we derive new results which allow to compare solution of two equations with different p and provide new integral characterization of the principal solution.

Keywords: half-linear differential equation, oscillation criteria, nonoscillation criteria, comparison theorems, Riccati equation, principal solution
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1. Introduction

In this paper we deal with the half-linear second order differential equation

$$(r(t)\Phi_p(x'))' + c(t)\Phi_p(x) = 0, \tag{1}$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$ on the interval I, which will be specified below.

The domain of the operator on the left hand side of (1) is defined to be the set of all continuous real-valued functions x defined on I such that x and $r\Phi_p(x')$ are continuously differentiable on I.

Equation (1) attracted big attention as an equation with one dimensional p-Laplacian. It turns out that there is a close relationship between (1) and radially symmetric solutions of PDE with p-Laplacian and $(p-1)$-degree power nonlinearity.

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Equation (1) is called half-linear, because a constant multiple of each solution is also a solution. If $p = 2$, then (1) reduces to linear equation, but the linearity is lost in the general case $p \neq 2$. For a comprehensive treatment focused on equation (1) and results up to year 2005 see [5].

The aim of this paper is to prove new comparison results for equation (1). In contrast to most other known comparison theorems, we do not compare equations with equal p. Another aim of this paper is to provide an alternative integral characterization of the principal solution of a nonoscillatory equation. To achieve these goals we use a modification of the so called linearization technique, see [3] and the reference therein.

The paper is organized as follows. In the next section we recall necessary elements of the oscillation theory for (1), in particular, the Riccati technique, and we derive inequalities which, in turn, are used as a main tool in the proofs of our main results in the next two sections. In Section 3 the inequalities are used to compare oscillatory properties of two half-linear equations, in Section 4 we study the concept of the so-called principal solution of (1) and its integral characterization.

2. Preliminaries

It is known that (1) can be studied using methods similar to those for the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \tag{2}$$

which is the special case of (1) for $p = 2$. In particular, the Sturmian theory for (2) extends almost verbatim to (1), see [5]. In the qualitative theory of half-linear differential equations, we study the problem of (non)existence of a positive solution of (1) on an unbounded or a bounded interval. This problem is connected with (non)oscillation or (dis)conjugacy of (1). Similarly to the linear case, equation (1) can be classified as *oscillatory* or *nonoscillatory* according to whether all nontrivial solutions of (1) have or do not have a sequence of zeros tending to infinity. Recall that equation (1) is said to be *disconjugate* on an interval I if every nontrivial solution of (1) has at most one zero on I and equation (1) is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that (1) is disconjugate on $[T, \infty)$.

If x is a solution of (1) having no zero on I, then one can verify that $w = r\Phi_p(x'/x)$ is a solution of the Riccati equation

$$R[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \tag{3}$$

where $q = \frac{p}{p-1}$ is the conjugate number to p. The existence of a solution of (1) having no zero on a compact or an open interval I (on an interval $[T, \infty)$) is guaranteed by disconjugacy of (1) on I (by nonoscillation of (1)). More precisely, the following statement holds.

Lemma A. ([5, Chapters 1.2 and 2.2]) *Suppose that I is either a compact or an open finite interval. The following statements are equivalent:*

(i) Equation (1) is disconjugate on I (nonoscillatory).
(ii) Equation (3) has a solution on I (on an interval $[T, \infty)$).
(iii) There exists a continuously differentiable function w such that $R[w] \le 0$ on I (on an interval $[T, \infty)$).

Our results are based on the following inequalities. These inequalities extend those in [2, Lemma 2.4].

Lemma 2.1. *Define the function*

$$P(a,b) := \frac{|a|^p}{p} - ab + \frac{|b|^q}{q}. \tag{4}$$

The following estimates hold.

(i) *Let $p \in (1,2]$. For every $\alpha \in [2,q]$ there exists a positive number $\beta_{\alpha,p}$ such that*
$$P(a,b) \ge \beta_{\alpha,p} |a|^{(p-1)(q-\alpha)} |b - \Phi_p(a)|^\alpha \tag{5}$$
for every a, $b \in \mathbb{R}$.

(ii) *Let $p \ge 2$. For every $\alpha \in [q,2]$ there exists a positive number $\beta_{\alpha,p}$ such that*
$$P(a,b) \le \beta_{\alpha,p} |a|^{(p-1)(q-\alpha)} |b - \Phi_p(a)|^\alpha \tag{6}$$
for every a, $b \in \mathbb{R}$, $a \ne 0$.

Proof. If $b = \Phi_p(a)$ then both (5) and (6) hold by direct computation. The function $P(a,b)$ is nonnegative by the Young inequality and hence the estimate (5) holds for $a = 0$.

Suppose that $a \ne 0$, $b \ne \Phi_p(a)$ and consider the function

$$Q(a,b) := \frac{P(a,b)}{|a|^{(p-1)(q-\alpha)} |b - \Phi_p(a)|^\alpha}.$$

It is sufficient to prove that the function Q is bounded below by a positive constant if $1 < p \le 2 \le \alpha \le q$ and bounded above if $1 < q \le \alpha \le 2 \le p$. Direct computation shows that $Q(a,b) = f(b/\Phi_p(a))$, where

$$f(x) = \frac{\frac{1}{p} - x + \frac{|x|^q}{q}}{|x-1|^\alpha}. \tag{7}$$

The function f is positive and continuous on $\mathbb{R} \setminus \{1\}$ and has the following properties

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} \infty & \text{for } \alpha < q, \\ \frac{1}{q} & \text{for } \alpha = q, \\ 0 & \text{for } \alpha > q, \end{cases}$$

$$\lim_{x \to 1} f(x) = \begin{cases} \infty & \text{for } \alpha > 2, \\ \frac{q-1}{2} & \text{for } \alpha = 2, \\ 0 & \text{for } \alpha < 2. \end{cases}$$

Now the statement of the lemma follows easily. $\square$

In the whole paper we suppose that the assumption

$$h \text{ is a positive differentiable function such that } h' \neq 0 \tag{8}$$

holds on the intervals under consideration. Following [3], denote

$$\begin{aligned} w_h(t) &= r(t)\Phi_p(h'(t)/h(t)), \\ G(t) &= h^p(t)w_h(t) = r(t)h(t)\Phi_p(h'(t)), \\ H(t,v) &= |v+G(t)|^q - q\Phi_q(G(t))v - |G(t)|^q, \end{aligned}$$

where $\Phi_q(x) = \Phi_p^{-1}(x) = |x|^{q-2}x$ is the inverse function to Φ_p.

Along with (3) consider the so-called modified Riccati equation

$$v' + C(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v) = 0, \tag{9}$$

where

$$C(t) = h(t)\Big[\big(r(t)\Phi_p(h'(t))\big)' + c(t)\Phi_p(h(t))\Big]. \tag{10}$$

Note that if $p = 2$, then (9) becomes the Riccati equation associated with the linear equation which results from (2) upon the transformation $x = hy$. In the nonlinear case, we have the following relation between the differential operators given in (3) and (9).

Lemma 2.2. *Put $v := h^p(w - w_h)$, where w is a continuously differentiable function. Then the following identity holds:*

$$h^p(t)R[w] = v' + C(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v). \tag{11}$$

In particular, if w is a solution of (3), then v is a solution of (9) and conversely, if v is a solution of (9), then $w = h^{-p}v + w_h$ is a solution of (3).

Proof. The proof is essentially a version of the proof of [3, Lemma 4] which is adjusted to our notation.

The definitions of the functions G and v imply (the dependence on t is suppressed in the notation)

$$\begin{aligned} v + G &= h^p w, \\ G' &= \big(r\Phi_p(h')\big)'h + r|h'|^p, \\ \Phi_q(G) &= r^{q-1}h^{q-1}h'. \end{aligned}$$

Differentiating $v = h^p w - G$ we get

$$v' = ph^{p-1}h'w + h^p w' - \big(r\Phi_p(h')\big)'h - r|h'|^p. \tag{12}$$

Similarly

$$\begin{aligned}H(t,v) &= |v+G|^q - q\Phi_q(G)v - |G|^q\\ &= |v+G|^q - q\Phi_q(G)h^p w + (q-1)|G|^q\\ &= h^{pq}|w|^q - qr^{q-1}h^{q-1}h'h^p w + (q-1)r^q h^q |h'|^p\end{aligned}$$

and hence

$$(p-1)r^{1-q}h^{-q}H(t,v) = (p-1)r^{1-q}h^p|w|^q - ph^{p-1}h'w + r|h'|^p. \tag{13}$$

Summing up (12) and (13) and adding the term $h^p c$ to both sides of the equation we get (11). □

The function $H(t,v)$ is closely connected with the function P defined in (4) by the relation

$$H(t,v) = qP(\Phi_q(G(t)), v + G(t)). \tag{14}$$

This means that $H(t,v)$ can be put under control using estimates from Lemma 2.1 as the following lemma shows.

Lemma 2.3. *The following inequalities hold for the function H:*

- (i) $H(t,v) \geq 0$.
- (ii) *Let $p \in (1,2]$. For every $\alpha \in [2,q]$ there exists a positive number $\beta_{\alpha,p}$ such that*

$$H(t,v) \geq q\beta_{\alpha,p}|G(t)|^{q-\alpha}|v|^\alpha \tag{15}$$

for every t, $v \in \mathbb{R}$.

- (iii) *Let $p \geq 2$. For every $\alpha \in [q,2]$ there exists a positive number $\beta_{\alpha,p}$ such that*

$$H(t,v) \leq q\beta_{\alpha,p}|G(t)|^{q-\alpha}|v|^\alpha \tag{16}$$

for every t, $v \in \mathbb{R}$.

Proof. The nonegativity of the function $H(t,v)$ follows from (14) and from the nonegativity of the function P.

For $p \in (1,2]$, $\alpha \in [2,q]$ the estimate (15) follows from (14), Lemma 2.1 and from the computation

$$P(\Phi_q(G(t)), v+G(t)) \geq \beta_{\alpha,p}|\Phi_q(G(t))|^{(p-1)(q-\alpha)}|v|^\alpha = \beta_{\alpha,p}|(G(t)|^{q-\alpha}|v|^\alpha.$$

In the case $p \geq 2$, $\alpha \in [q,2]$ we have the opposite inequality which proves (16). □

Remark 2.1. Note that the constant $\beta_{\alpha,p}$ can be computed analyticaly if $\alpha = 2$. In this case $\beta_{2,p} = \frac{1}{2}$, see [2, Lemma 2.4]. For general α we can obtain $\beta_{\alpha,p}$ as the supremum or infimum of the function Q defined in the proof of Lemma 2.1. It can be also proved that both (5) and (6) are valid with $\beta_{\alpha,p} = 4\alpha^{-1}2^{-\alpha}$. This gives constant $\beta_{\alpha,p}$ which is not optimal like the supremum or infimum of the function Q, but this constant is computed explicitly and does not depend on p. Really, dividing (5) by $|a|^p$ and substituting $x = b/\Phi_p(a)$, $\beta_{\alpha,p} = 4\alpha^{-1}2^{-\alpha}$ inequality (5) transforms into

$$\frac{1}{p} - x + \frac{|x|^q}{q} \geq \frac{4}{\alpha 2^\alpha}|x-1|^\alpha.$$

Inequality (6) transforms similarly into an opposite inequality. The fact that these inequalities are valid for $p \in (1,2]$ and $p \geq 2$ respectively has been proved in [4, Lemma 2.1].

3. Comparison with respect to p

There are only few results related to comparison of two half-linear differential equations with different power in the nonlinearity, see [5, Theorem 2.3.5], [9] and [10, Theorem 4.1]. The results show that bigger power in nonlinearity speeds up the oscillation, similarly like bigger coefficient $c(t)$ or smaller coefficient $r(t)$ in (1).

Here we use the approach based on comparison of the Riccati equations related with two half-linear equations with different power in the nonlinearity. The connection between these two Riccati equations is arranged by the modified Riccati equation (9), using identity (11) and inequalities (15), (16). Similar idea has been used in the series of papers dealing with the case when $\alpha = 2$ in (15), (16), i.e., equation (1) is compared with a certain associated linear equation of the form (2), see [3] and the references given therein.

In this section we suppose that (8) holds on the interval under consideration and let $\alpha, \beta_{\alpha,p}$ be real constants from Lemma 2.1 (Lemma 2.3), that is,

- if $p \in (1,2]$, then $\alpha \in [2,q]$ and $\beta_{\alpha,p} > 0$ is such that (5), (15) hold,
- if $p \geq 2$, then $\alpha \in [q,2]$ and $\beta_{\alpha,p} > 0$ is such that (6), (16) hold.

Denote $\alpha^* = \frac{\alpha}{\alpha-1}$ the conjugate number to α and consider the half-linear differential equation

$$(R_{\alpha^*}(t)\Phi_{\alpha^*}(x'))' + C(t)\Phi_{\alpha^*}(x) = 0, \quad \Phi_{\alpha^*}(x) := |x|^{\alpha^*-2}x, \tag{17}$$

where $C(t)$ is defined in (10) and

$$R_{\alpha^*}(t) = \left[\frac{p\beta_{\alpha,p}}{\alpha^*-1}\right]^{1-\alpha^*} r(t)h^{\alpha^*}(t)|h'(t)|^{p-\alpha^*}. \tag{18}$$

Note that the corresponding Riccati equation is

$$v' + C(t) + (\alpha^*-1)R_{\alpha^*}^{1-\alpha}|v|^\alpha = 0$$

and using the definition of the function R_{α^*} we see that this equation can be written in the form

$$v' + C(t) + p\beta_{\alpha,p}r^{1-\alpha}(t)h^{-\alpha}(t)|h'(t)|^{(p-1)(q-\alpha)}|v|^\alpha = 0. \tag{19}$$

Theorem 3.1. *Let $p \geq 2$ and $\alpha \in [q, 2]$.*

(i) *Suppose that I is either a compact or an open finite interval and that (8) holds on I. If (17) is disconjugate on the interval I, then (1) is also disconjugate on I.*

(ii) *Suppose that (8) holds for large t. If (17) is nonoscillatory, then (1) is also nonoscillatory.*

PROOF. (i) The assumptions imply that (19) has a solution v defined on I. From (16) we have

$$\begin{aligned}(p-1)&r^{1-q}(t)h^{-q}(t)H(t,v)\\ &\leq (p-1)r^{1-q}(t)h^{-q}(t)q\beta_{\alpha,p}r^{q-\alpha}(t)h^{q-\alpha}(t)|h'(t)|^{(p-1)(q-\alpha)}|v|^{\alpha}\\ &= p\beta_{\alpha,p}r^{1-\alpha}(t)h^{-\alpha}(t)|h'(t)|^{(p-1)(q-\alpha)}|v|^{\alpha}.\end{aligned} \tag{20}$$

Now Lemma 2.2 implies that $w = h^{-p}v + w_h$ is a solution of $R[w] \leq 0$ on the interval I. Hence, by Lemma A, equation (1) is disconjugate on I.

(ii) If (17) is nonoscillatory, then, by Lemma A, there exists a number T such that (19) has a solution v on $[T, \infty)$. Now, using the same arguments as in the part (i) we prove that there exists a solution w such that $R[w] \leq 0$ on $[T, \infty)$, and hence (1) is nonoscillatory by Lemma A. □

Theorem 3.2. *Let $p \in (1, 2]$ and $\alpha \in [2, q]$.*

(i) *Suppose that I is either a compact or an open finite interval and that (8) holds on I. If (1) is disconjugate on I, then (17) is also disconjugate on I.*

(ii) *Suppose that (8) holds for large t. If (1) is nonoscillatory, then (17) is also nonoscillatory.*

PROOF. The proof is similar to that of Theorem 3.1. The discongugacy of (1) on I (nonoscillation of (1)) implies the existence of a solution w of (3) on I (on $[T, \infty)$), i.e., in view of Lemma 2.2, $v = h^p(w - w_h)$ is a solution of (9) on I (on $[T, \infty)$). From Lemma 2.3 we get the opposite inequality to (20), which implies that v solves the inequality

$$v' + C(t) + p\beta_{\alpha,p}r^{1-\alpha}(t)h^{-\alpha}(t)|h'(t)|^{(p-1)(q-\alpha)}|v|^{\alpha} \leq 0$$

on I (on $[T, \infty)$). The disconjugacy (nonoscillation) of (17) follows from Lemma A. □

4. Integral characterization of the principal solution

The concept of the principal solution of (1) has been introduced in [8] via the minimal solution of the associated Riccati equation (3). If (1) is nonoscillatory, then there exists a solution of (3) which is extensible up to infinity. It was shown in [8], that among all solutions of (3) with this property there exists the so-called *minimal solution* $\tilde{w}$, which is minimal in the following sense. If $\tilde{w}$ and w are two distinct solutions of (3) defined on $[T, \infty)$, then $w(t) > \tilde{w}(t)$ for $t \in [T, \infty)$.

The *principal solution* $\tilde{x}$ of (1) is defined as the solution which determines the minimal solution $\tilde{w}$ of (3) via the substitution $\tilde{w} = r\Phi_p(\tilde{x}'/\tilde{x})$, i.e.,

$$\tilde{x}(t) = C \exp\left\{ \int^t \Phi_q\big(\tilde{w}(s)/r(s)\big) \, \mathrm{d}s \right\}.$$

Note that the principal solution was introduced independently of [8] in [7] using the half-linear Prüfer transformation.

A well-know result from the theory of linear differential equations states, that a solution $\tilde{x}$ of the second order linear differential equation (2) is principal if and only if the integral $\int^\infty \frac{\mathrm{d}t}{r(t)\tilde{x}^2(t)}$ is divergent. However, this integral characterization of the principal solution is no more valid for half-linear differential equations. A proper extension of this property is established in [2]. In particular, the following theorem holds.

Theorem B ([2, Theorems 3.1 and 3.2]). *Suppose that* (1) *is nonoscillatory and $\tilde{x}$ is its solution which satisfies $\tilde{x}'(t) \neq 0$ for large t.*

(i) *Let $p \geq 2$. If $\tilde{x}$ is a principal solution, then*

$$\int^\infty \frac{\mathrm{d}t}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \infty \tag{21}$$

holds.

(ii) *Let $p \in (1, 2]$. If* (21) *holds, then $\tilde{x}$ is a principal solution.*

Note that equivalent integral characterization of the principal solution is lost in Theorem B, see Example 4.1 below. Several attempts have been made to find an alternative and more general integral characterization, see [1, 6]. See [1] for the survey of the known results.

Using the inequalities established in Lemma 2.3 we can offer an alternative integral property which extends Theorem B in another direction than those in the papers mentioned above. This approach leads to the following theorem.

Theorem 4.1. *Suppose that* (1) *is nonoscillatory and $h(t)$ is its positive solution which satisfies $h'(t) \neq 0$ for large t.*

(i) *Let $p \geq 2$. If h is a principal solution, then for every $\alpha \in [q, 2]$*

$$\int^\infty \frac{\mathrm{d}t}{r^{\alpha-1}(t)h^\alpha(t)|h'(t)|^{(p-1)(\alpha-q)}} = \infty \tag{22}$$

holds.

(ii) *Let $p \in (1, 2]$. If* (22) *holds for some $\alpha \in [2, q]$, then h is a principal solution.*

Proof. (i) Suppose, by contradiction, that there exists $\alpha \in [q, 2]$ such that (22) does not hold. By Lemma 2.3 there is $\beta_{\alpha,p} > 0$ such that (16) holds. Suppose that T is so large that

$$\int_T^\infty \frac{\mathrm{d}t}{r^{\alpha-1}(t)h^\alpha(t)|h'(t)|^{(p-1)(\alpha-q)}} < \frac{1}{2\beta_{\alpha,p}(\alpha-1)p}. \tag{23}$$

Denote $w_h(t) = r(t)\Phi_p\Big(h'(t)/h(t)\Big)$ and consider the solution w of (3) satisfying the initial condition $w(T) = w_h(T) - h^{-p}(T)$. By this definition we have $w(T) < w_h(T)$ and the unique solvability of (3) implies that $w(t) < w_h(t)$ for all $t \in [T, T^*)$, where $[T, T^*)$ is the maximal interval of existence of the solution w.

From (10) we have $C \equiv 0$ and the function $v = h^p(w - w_h)$ is a solution of

$$v' + (p-1)r^{1-q}h^{-q}H(t, v) = 0 \tag{24}$$

by Lemma 2.2. This solution satisfies $v(T) = -1$ and $v(t) < 0$ for all t for which w (and hence also v) is defined, i.e., for $t \in [T, T^*)$.

Since $p \geq 2$ and $\alpha \in [q, 2]$, inequality (16) implies (20), i.e.,

$$(p-1)r^{1-q}h^{-q}H(t, v) \leq p\beta_{\alpha,p}r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}|v|^\alpha.$$

Hence v solves the inequality

$$v' + p\beta_{\alpha,p}r^{1-q}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}|v|^\alpha \geq 0$$

on $[T, T^*)$ and consequently

$$-\frac{v'}{|v|^\alpha} \leq p\beta_{\alpha,p}r^{1-q}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}.$$

Integrating this inequality over $[T, t]$, where $t \in (T, T^*)$, and using (23) we obtain

$$\begin{aligned}
\frac{1}{(\alpha-1)|v(T)|^{\alpha-1}} - \frac{1}{(\alpha-1)|v(t)|^{\alpha-1}} &\leq p\beta_{\alpha,p}\int_T^t r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}\,\mathrm{d}t \\
&\leq p\beta_{\alpha,p}\int_T^\infty r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}\,\mathrm{d}t \\
&< \frac{1}{2(\alpha-1)}.
\end{aligned}$$

From here and from $v(T) = -1$ we get

$$\frac{1}{(\alpha-1)|v(t)|^{\alpha-1}} > \frac{1}{\alpha-1} - \frac{1}{2(\alpha-1)} = \frac{1}{2(\alpha-1)},$$

and hence

$$|v(t)| < 2^{1/(\alpha-1)}.$$

This means that $v(t)$ is continuable up to infinity ($T^* = \infty$). Hence $v(t) < 0$ for all $t \geq T$, i.e. $w(t) < w_h(t)$ for all $t \geq T$. This shows that w_h is not the minimal solution of (3) and hence h is not the principal solution of (1).

(ii) Suppose, by contradiction, that (22) holds for some $\alpha \in [2, q]$ and h is not principal. Then $w_h = r\Phi_p(h'/h)$ is not minimal solution of (3) and hence there exists a solution w of (3) and a number T such that $w(t) < w_h(t)$ for $t \geq T$.

From identity (11) we see that the function $v := h^p(w - w_h)$ is a solution of equation (24). Since $p \in (1, 2]$ and $\alpha \in [2, q]$, by Lemma 2.3 there exists $\beta_{\alpha,p} > 0$ such that (15) and consequently also

$$(p-1)r^{1-q}h^{-q}H(t,v) \geq p\beta_{\alpha,p}r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}|v|^{\alpha}$$

holds on $[T, \infty)$. This means that v is a solution of the inequality

$$v' + p\beta_{\alpha,p}r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}|v|^{\alpha} \leq 0,$$

which can be rewritten as

$$-\frac{v'}{|v|^{\alpha}} \geq p\beta_{\alpha,p}r^{1-\alpha}h^{-\alpha}|h'|^{(p-1)(q-\alpha)}. \tag{25}$$

The inequality $w(t) < w_h(t)$ for $t \geq T$ implies $v(t) < 0$ for $t \geq T$ and integration of (25) over the interval $[T, t]$ gives

$$\frac{1}{(\alpha-1)|v(T)|^{\alpha-1}} - \frac{1}{(\alpha-1)|v(t)|^{\alpha-1}} \geq p\beta_{\alpha,p}\int_T^t r^{1-\alpha}(s)h^{-\alpha}(s)|h'(s)|^{(p-1)(q-\alpha)}\,\mathrm{d}s.$$

From here, letting $t \to \infty$, we have

$$\frac{1}{(\alpha-1)|v(T)|^{\alpha-1}} \geq p\beta_{\alpha,p}\int_T^{\infty} r^{1-\alpha}(s)h^{-\alpha}(s)|h'(s)|^{(p-1)(q-\alpha)}\,\mathrm{d}s,$$

which contradicts (22). □

Remark 4.1.
- It is easy to see that if $\alpha = 2$, then Theorem 4.1 reduces to Theorem B.
- In the linear case we have $p = q = 2$ and the interval for α shrinks to the single point $\alpha = 2$. Hence Theorem 4.1 does not introduce anything new in the linear case. The following example shows that our extension to general α is not dummy in the general half-linear case.

Example 4.1. Consider equation

$$(\Phi_{3/2}(x'))' + \frac{15t^{-3/2}}{(t^9-1)^{1/2}}\Phi_{3/2}(x) = 0, \quad t > 1. \tag{26}$$

In this setting we have $p = 3/2$, $q = 3$, $r(t) = 1$ and $c(t) = 15t^{-3/2}/(t-1)^{1/2}$. This equation has a solution $h(t) = 1 - 1/t^9$. Direct computation shows

$$\int^{\infty} \frac{\mathrm{d}t}{r^{\alpha-1}(t)h^{\alpha}(t)|h'(t)|^{(p-1)(\alpha-q)}} = \int^{\infty} \frac{\mathrm{d}t}{3^{\alpha-3}(1-t^{-9})^{\alpha}t^{15-5\alpha}}$$
$$= \int^{\infty} \frac{\mathrm{d}t}{3^{\alpha-3}t^{15-5\alpha}}$$

and the integral diverges for $\alpha \in [14/5, 3]$. Hence, by Theorem 4.1, h is a principal solution of (26), whereas Theorem B ($\alpha = 2$) fails.

Note that equation (26) is taken from [1], where the principality of h has been already mentioned using another argument. However, the proof of the principality in [1] heavily depends on the properties of the linearly independent solutions which is known in some sense (from the fact that the quasiderivative tends to zero we know, that the linearly independent solutions are unique up to a constant multiple). With Theorem 4.1 we do not need any additional information on linearly independent solutions.

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