A remark on connection between conjugacy of half-linear differential equation and equation with mixed nonlinearities☆

Robert Mařík

Mendel University, Department of Mathematics
Zemědělská 3, 613 00 Brno, Czech Republic

Abstract

In the paper new criteria for conjugacy of half-linear ordinary differential equation are derived by using Riccati transformation. These criteria are used to derive nonexistence and oscillation results for equation with mixed nonlinearities, which is viewed as a perturbation of half-linear equation.

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1. Introduction

In the paper we study equation

$$(r(t)|x'|^{p-2}x')' + c(t)|x|^{p-2}x + \sum_{i=1}^{m} c_i(t)|x|^{p_i-2}x = e(t), \tag{1.1}$$

where $p > 1$ and $p_i > 1$ are real numbers, $c(t)$, $c_i(t)$ and $e(t)$ are continuous functions and $r(t)$ is a positive continuous function. Under solution of (1.1) on the interval I we understand a smooth function $x(t)$ defined on I such that $r(t)|x'(t)|^{p-2}x'(t)$ is differentiable and $x(t)$ satisfies (1.1) everywhere on I. We suppose that $p_i \neq p$ for every i and $p_i \neq p_j$ for every i, j with property $i \neq j$.

The paper is motivated by recent paper [5] and extends and completes the results from this paper in several respects (see also Remark 2.2). In contrast to the paper [5] we allow $p_i < p$ for some i and do not assume anything about the fixed sign of the functions c_i in this case.

The oscillation of the half-linear equation has been studied using generalized Riccati substitution in [3, Theorem 2] and the following theorem has been proved.

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Email address: `marik@mendelu.cz` (Robert Mařík)

Theorem A (Li, Cheng, [3]). *Suppose that for any $T \geq t_0$ there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) < 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \neq 0, t \in (s_i, t_i)$ and $u(s_i) = 0 = u(t_i)\}$ for $i = 1,\ 2$. If there exist $H \in D(s_i, t_i)$ and a positive nondecreasing function $\phi \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\int_{s_i}^{t_i} H^2(t)\phi(t)C(t)\,\mathrm{d}r > \frac{1}{p^p}\int_{s_i}^{t_i} \frac{r(t)\phi(t)}{|H(t)|^{p-2}}\left(2|H'(t)| + |H(t)|\frac{\phi'(t)}{\phi(t)}\right)^p \mathrm{d}t\,, \tag{1.2}$$

for $i = 1, 2$, then

$$(r(t)|x'|^{p-2}x')' + C(t)|x|^{p-2}x = 0 \tag{1.3}$$

is oscillatory.

As pointed out in [5], this result cannot be applied if $p > 2$. From this reason Han, Wang and Zheng presented in [5] an extension of this theorem which removed the restriction $p \leq 2$ and also extends this theorem for equation with mixed nonlinearities. However, the results from [5] do not include Theorem A as a special case.

In this paper we present another extension of Theorem A which also removes the restriction $p \leq 2$ and in contrast to [5] includes both results from [3, 5] as a special case and deals with more general equation. Instead of to formulate the results as oscillation criteria which are in fact consequence of conjugacy criteria, we present our results in terms of nonexistence of positive and negative solutions. The extension to oscillation criterion is trivial and straightforward. This idea is also motivated by the fact that as far as the author knows, we miss systematic oscillation theory for the equation with mixed nonlinearities. The reason of this lack is in the fact that the set of all solutions is more comprehensive than in the half-linear case. In particular, the solution may become infinite at some finite t (see [4] for more detailed discussion).

Recall that t_1, $t_2 \in I$ are said to be *conjugate* point relative to Eq. (1.3) if there exists a nontrivial solution $x(t)$ of this equation which satisfies $x(t_1) = 0 = x(t_2)$. Since oscillation theory attracts more attention than problems related to conjugacy, the literature related to oscillation is much more comprehensive. However many oscillation and nonoscillation criteria are in fact conjugacy or disconjugacy criteria in a neighborhood of infinity (see [1] for some recent progress in this field) or on a sequence of intervals tending to infinity (see Theorem A).

We adopt the main idea of the paper [5] and we will consider Eq. (1.1) as a perturbation of half-linear differential equation (1.3). In contrast to [5] we do not use the generalized Riccati transformation

$$w(t) = \phi(t)r(t)\frac{|x'(t)|^{p-2}x'(t)}{|x(t)|^{p-2}x(t)},$$

but we consider the special case $\phi(x) = 1$, i.e. we use the transformation

$$w(t) = r(t)\frac{|x'(t)|^{p-2}x'(t)}{|x(t)|^{p-2}x(t)}, \tag{1.4}$$

which converts Eq. (1.3) into

$$w' = -(p-1)r^{1-q}(t)|w|^q - C(t). \tag{1.5}$$

There is no loss of generality in this approach, since the results from [5] can be obtained from (1.5) by transformation. As an advantage, some intermediate calculations like proof of Theorem 2.1 are simpler and more transparent.

2. Main results

The following lemma is used to estimate terms involving powers α with term with power β. This estimate is necessary to collect all terms into a term with power $p-1$. In contrast to [5] we allow $\alpha < \beta$.

Lemma 2.1. *The following inequalities hold for $a \geq 0$ and $x > 0$.*

1. *If $\alpha < \beta$ and $b > 0$, then $b - ax^\alpha \geq -x^\beta \left(\frac{a(\beta-\alpha)}{\beta}\right)^{\frac{\beta}{\alpha}} \frac{\alpha}{\beta-\alpha} b^{1-\frac{\beta}{\alpha}}$.*
2. *If $\alpha > \beta$ and $b \geq 0$, then $ax^\alpha + b \geq x^\beta \left(\frac{a(\alpha-\beta)}{\beta}\right)^{\frac{\beta}{\alpha}} \frac{\alpha}{\alpha-\beta} b^{1-\frac{\beta}{\alpha}}$.*

Proof. Divide both inequalities by x^β. Now the inequalities can be proved directly by inspecting functions which appears on the left hand sides. □

In the following theorem $[c_i(t)]_+ = \max\{c_i(t), 0\}$ denotes the positive part of the function $c_i(t)$.

Theorem 2.1. *Let $e(t) < 0$ on $[a,b]$ and denote*

$$\begin{aligned} C(t) = c(t) &+ \sum_{i \in I_1} \frac{p_i - 1}{p_i - p} \left[\frac{c_i(t)(p_i - p)}{p-1}\right]^{(p-1)/(p_i-1)} \left(\varepsilon_i |e(t)|\right)^{\frac{p_i - p}{p_i - 1}} \\ &- \sum_{i \in I_2} \frac{p_i - 1}{p - p_i} \left[\frac{[-c_i(t)]_+(p - p_i)}{p-1}\right]^{(p-1)/(p_i-1)} \left(\varepsilon_i |e(t)|\right)^{\frac{p_i - p}{p_i - 1}}, \end{aligned} \tag{2.1}$$

where $I_1 = \{i \in [1,m] \cap \mathbb{N} : p_i > p\}$, $I_2 = \{i \in [1,m] \cap \mathbb{N} : p_i < p\}$, $\varepsilon_i > 0$, $\sum_{i=1}^m \varepsilon_i = 1$.

If Eq. (1.3) has conjugate points on $[a,b]$, then Eq. (1.1) has no positive solution on $[a,b]$.

Moreover, if $I_2 = \emptyset$, then the inequality $e(x) < 0$ can be relaxed to $e(x) \leq 0$.

Proof. Suppose that x is a positive solution of (1.1) on $[a,b]$ and let the function w be defined by the Riccati substitution (1.4). Differentiating (1.4) and using

(1.1) we get

$$
\begin{aligned}
w'(t) &= (1-p)|w(t)|^q r^{1-q}(t) \\
&\quad - c(t) - \sum_{i=1}^{m} c_i(t) x^{p_i-p}(t) + \frac{e(t)}{x^{p-1}(t)} \\
&= (1-p)|w(t)|^q r^{1-q}(t) \\
&\quad - c(t) - \frac{1}{x^{p-1}(t)} \sum_{i=1}^{m} \Big(c_i(t) x^{p_i-1}(t) - \varepsilon_i e(t) \Big).
\end{aligned}
\tag{2.2}
$$

If $p_i > p$, then using part (2) of Lemma 2.1 we have

$$
\begin{aligned}
c_i(t)x^{p_i-1}(t) - \varepsilon_i e(t) &= c_i(t)x^{p_i-1}(t) + \varepsilon_i |e(t)| \\
&\geq x^{p-1}(t) \left[\frac{c_i(t)(p_i-p)}{p-1} \right]^{(p-1)/(p_i-1)} \frac{p_i-1}{p_i-p} \big(\varepsilon_i |e(t)| \big)^{\frac{p_i-p}{p_i-1}}.
\end{aligned}
$$

If $p_i < p$, then using part (1) of Lemma 2.1 we have

$$
\begin{aligned}
c_i(t)x^{p_i-1}(t) - \varepsilon_i e(t) &= \varepsilon_i |e(t)| - (-c_i(t)) x^{p_i-1}(t) \\
&\geq \varepsilon_i |e(t)| - [-c_i(t)]_+ x^{p_i-1}(t) \\
&\geq -x^{p-1}(t) \left[\frac{[-c_i(t)]_+(p-p_i)}{p-1} \right]^{(p-1)/(p_i-1)} \frac{p_i-1}{p-p_i} \big(\varepsilon_i |e(t)| \big)^{\frac{p_i-p}{p_i-1}}.
\end{aligned}
$$

Summing up the last two estimates over all $i \in I_1$ and $i \in I_2$, respectively, dividing by $|x|^{p-1}$ and using definition of $C(t)$ we get

$$
C(t) - c(t) \leq \frac{1}{x^{p-1}(t)} \sum_{i=1}^{m} \Big(c_i(t)x^{p_i-1}(t) - \varepsilon_i e(t) \Big)
$$

and from (2.2) it follows that

$$
w'(t) \leq (1-p)|w(t)|^q r^{1-q}(t) - C(t)
$$

holds on $[a, b]$. Using simple comparison argument or using [2, Theorem 2.2.1] it can be shown, that the generalized Riccati equation

$$
v'(t) = (1-p)|v(t)|^q r^{1-q}(t) - C(t)
$$

has solution on $[a, b]$. Hence by half-linear Roundabout theorem (see [2]), Eq. (1.3) has no conjugate points on $[a, b]$. Theorem is proved. □

Remark 2.1. Note that we have no sign restriction on the functions $c_i(t)$ if $p_i < p$ and the negative parts of the functions $c_i(t)$ play a role in the function $C(t)$.

Corollary 2.1. *Theorem 2.1 remains valid, if we replace the condition $e(t) < 0$ ($e(t) \leq 0$) by $e(t) > 0$ ($e(t) \geq 0$) and the words "positive solution" by "negative solution".*

Proof. Follows from the fact that if $x(t)$ is a solution of (1.1), then $(-x(t))$ is a solution of equation in the same form but with the right-hand side $(-e(t))$. $\square$

Theorem 2.2. *Suppose that there exist a real number α, $\alpha \geq p$ and smooth functions h, ϕ, such that $h(a) = 0 = h(b)$, $h(t) > 0$ on (a, b), ϕ is positive on $[a, b]$ and*

$$\int_a^b h^\alpha(t)\phi(t)C(t)\,\mathrm{d}t > \frac{1}{p^p}\int_a^b \left|\alpha h'(t) + h(t)\frac{\phi'(t)}{\phi(t)}\right|^p r(t)\phi(t)h^{\alpha-p}(t)\,\mathrm{d}t\,. \tag{2.3}$$

Then Eq. (1.3) has conjugate points on $[a, b]$.

Proof. Suppose that Eq. (1.3) has no conjugate points on $[a, b]$. Then there exists a positive solution $x(t)$ of this equation on $[a, b]$ and the Riccati type transformation (1.4) defines a function $w(t)$ which solves the Riccati type equation (1.5) on $[a, b]$. The function $W(t) = \phi(t)w(t)$ satisfies

$$W'(t) = \frac{\phi'(t)}{\phi(t)}W(t) + (1-p)|W(t)|^q r^{1-q}(t)\phi^{1-q}(t) - \phi(t)C(t)$$

on $[a, b]$. Rearranging terms, multiplying by $h^\alpha(t)$ and integrating over the interval $[a, b]$ we get

$$\begin{aligned}\int_a^b h^\alpha(t)\phi(t)C(t)\,\mathrm{d}t &= -\int_a^b h^\alpha(t)W'(t)\,\mathrm{d}t + \int_a^b h^\alpha(t)\frac{\phi'(t)}{\phi(t)}W(t)\,\mathrm{d}t\\ &\quad - (p-1)\int_a^b h^\alpha(t)r^{1-q}(t)\phi^{1-q}(t)|W(t)|^q\,\mathrm{d}t\,.\end{aligned}$$

Integrating by parts and using the conditions $h(a) = 0 = h(b)$ we get

$$-\int_a^b h^\alpha(t)W'(t)\,\mathrm{d}t = \alpha\int_a^b h^{\alpha-1}(t)h'(t)W(t)\,\mathrm{d}t\,.$$

Hence

$$\begin{aligned}\int_a^b h^\alpha(t)\phi(t)C(t)\,\mathrm{d}t &\leq \int_a^b \left|\alpha h^{\alpha-1}(t)h'(t) + h^\alpha(t)\frac{\phi'(t)}{\phi(t)}\right||W(t)|\,\mathrm{d}t\\ &\quad - (p-1)\int_a^b h^\alpha(t)r^{1-q}(t)\phi^{1-q}(t)|W(t)|^q\,\mathrm{d}t\,.\end{aligned} \tag{2.4}$$

Since the Young inequality implies $A|W| - (p-1)B|W|^q \leq \frac{1}{p^p}A^pB^{1-p}$, we get the following estimate on $[a, b]$

$$\begin{aligned}|W(t)|&\left|\alpha h^{\alpha-1}(t)h'(t) + h^\alpha(t)\frac{\phi'(t)}{\phi(t)}\right| - (p-1)h^\alpha(t)r^{1-q}(t)\phi^{1-q}(t)|W(t)|^q\\ &\leq \frac{1}{p^p}\left|\alpha h'(t) + h(t)\frac{\phi'(t)}{\phi(t)}\right|^p r(t)\phi(t)h^{\alpha-p}(t).\end{aligned}$$

Integrating over (a, b) and using this estimate in (2.4) we get an inequality which contradicts (2.3). $\square$

Summarizing Theorem 2.1 and 2.2 we get the following oscillation result. Recall that, adopting terminology of [5], Eq. (1.1) is said to be oscillatory if all its nontrivial solutions (i.e. the solutions extensible up to infinity which are not identically equal to zero in a neighborhood of infinity) have arbitrarily large zeros.

Corollary 2.2. *Assume that for every $T \geq t_0$ there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) < 0$ for $t \in [s_1, t_1]$ and $e(t) > 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) > 0 \text{ for } t \in (s_i, t_i) \text{ and } u(t_i) = 0 = u(s_i)\}$ for $i = 1, 2$. Let C be defined by (2.1). If there exists $H \in D(s_i, t_i)$ and a positive function ϕ such that (2.3) holds with $a = s_i$, $b = t_i$ for $i = 1, 2$, then Eq. (1.1) is oscillatory. Moreover, if $p_i > p$ for every i, then the inequalities $e(t) < 0$ and $e(t) > 0$ can be relaxed to $e(t) \leq 0$ and $e(t) \geq 0$, respectively.*

Proof. Using Theorems 2.1 and 2.2 on the interval $[s_1, t_1]$ we can see that there is no positive solution on $[s_1, t_1]$ and thus there is no positive solution on $[s_1, t_2]$. Taking into account Corollary 2.1 we can prove in a similar way that there exists no negative solution on $[s_2, t_2]$ and thus there is no negative solution on $[s_1, t_2]$. □

Remark 2.2. The main result of the paper [5] is a special case of Corollary 2.2. Really, if we require $p_i > p$ (and thus also $c_i(t) \geq 0$ on (s_i, t_i)) for every i and if we put $\alpha = p$ and put $\varepsilon_i = \frac{1}{m}$, then inequality (2.3) in Corollary 2.2 implies (12) of [5, Theorem 2.2]. Since s_1 can be arbitrarily large, Eq. (1.1) is oscillatory. Remark also that we use more accurate estimates when handling absolute values and thus (2.3) is less restrictive than (12) of [5] and thus yields more general criterion. Remark also that Theorem A is another special case of Corollary 2.2 (for $\alpha = 2$ and $c_i \equiv 0$).

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