

Funkce $f(x,y)$ splňuje

$$f(3,1) = 7$$

$$\frac{\partial f}{\partial x}(3,1) = 5$$

$$\frac{\partial f}{\partial y}(3,1) = -2$$

Najděte lineární aproximaci funkce $f(x,y)$
v okolí bodu $(3,1)$.

$$f(x,y) \approx 7 + 5(x-3) + (-2)(y-1)$$

$$f(x,y) \approx 7 + 5(x-3) - 2(y-1)$$

(Vše potřebné pro dosazení do vzorce je
samočíslo zadání, nic navíc potřeba
dopočítávat.)

Najdi lineární aproximaci funkce $f(x,y)$ v bodě
bodu (x_0, y_0)

$$1) f(x,y) = x \cdot \sqrt{x+y} \quad (x_0, y_0) = (3, 1)$$

$$\nabla f = \left(\sqrt{x+y} + x \cdot \frac{1}{2}(x+y)^{-1/2}, x \cdot \frac{1}{2}(x+y)^{-1/2} \right)$$

$$\begin{aligned} \nabla f(3,1) &= \left(\sqrt{4} + 3 \cdot \frac{1}{2}(4)^{-1/2}, 3 \cdot \frac{1}{2}(4)^{-1/2} \right) = \\ &= \left(2 + \frac{3}{4}, \frac{3}{4} \right) = \left(\frac{11}{4}, \frac{3}{4} \right) \end{aligned}$$

$$f(3,1) = 3 \cdot \sqrt{4} = 6$$

$$\begin{aligned} f(x,y) &\approx f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x-x_0, y-y_0) = \\ &= 6 + \left(\frac{11}{4}, \frac{3}{4} \right) \cdot (x-3, y-1) = \\ &= 6 + \frac{11}{4}(x-3) + \frac{3}{4}(y-1) \end{aligned}$$

$$2) f(x,y) = \frac{xy^2}{x+1} \quad (x_0, y_0) = (1, 2)$$

$$f(x,y) = \frac{x}{x+1} \cdot y^2$$

$$\frac{\partial f}{\partial x} = y^2 \frac{1 \cdot (x+1) - x \cdot 1}{(x+1)^2} = y^2 \frac{1}{(x+1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x}{x+1} \cdot 2y$$

$$\begin{aligned} \nabla f(1,2) &= \left(4 \cdot \frac{1}{(1+1)^2}, \frac{1}{1+1} \cdot 4 \right) = \\ &= (1, 2) \end{aligned}$$

$$f(1,2) = \frac{1 \cdot 4}{1+1} = 2$$

$$\begin{aligned} f(x,y) &\approx 2 + (1, 2)(x-1, y-2) = \\ &= 2 + (x-1) + 2(y-2) \end{aligned}$$

Najděte řešení k funkci dani implicitně

rovnice

$$x + y = 2x \cdot e^y - 1$$

→ bodem odvozuje $(1, 0)$.

$$f(x, y) = x + y - 2x e^y + 1$$

$$\nabla f = (1 - 2e^y, 1 - 2x e^y)$$

$$\nabla f(1, 0) = (-1, -1)$$

tečna: $\nabla f(1, 0) \cdot (x - 1, y - 0) = 0$

$$-(x - 1) - (y - 0) = 0$$

$$1 - x - y = 0$$

Najděte tečnu k funkci danoj implicitně
rovnici

$$3x + y = \sqrt{y} + \ln(x) + 5$$

↳ bodem dotyku $(1, 4)$.

$$f(x, y) = 3x + y - \sqrt{y} - \ln(x) - 5$$

$$\nabla f = \left(3 - \frac{1}{x}, 1 - \frac{1}{2\sqrt{y}} \right)$$

$$\nabla f(1, 4) = \left(2, \frac{3}{4} \right)$$

$$\nabla f(1, 4) \cdot ((x-1), (y-4)) = 0$$

$$\left(2, \frac{3}{4} \right) \cdot (x-1, y-4) = 0$$

$$2(x-1) + \frac{3}{4}(y-4) = 0$$

$$8(x-1) + 3(y-4) = 0$$

nebo (po rozkrojení)

$$8x + 3y - 20 = 0$$

Najděte funkci φ takovou, že gradientem této funkce je vektorové pole

$$\vec{F} = (xy^4 + 3x^2) \vec{i} + (2x^2y^3 + 4y) \vec{j}$$

Označme: $M = xy^4 + 3x^2$ $N = 2x^2y^3 + 4y$

Kontrola: $\frac{\partial M}{\partial y} = 4xy^3$ $\frac{\partial N}{\partial x} = 4xy^3$

Hledáme φ takovou, že $\frac{\partial \varphi}{\partial x} = M$ a $\frac{\partial \varphi}{\partial y} = N$

$$\begin{aligned} \varphi &= \int M dx = \int (xy^4 + 3x^2) dx = \\ &= \underline{\underline{\frac{1}{2}x^2y^4 + x^3 + C(y)}} \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial \varphi}{\partial y} &= 2x^2y^3 + 0 + C'(y) \\ N &= 2x^2y^3 + 4y \end{aligned} \right\} \Rightarrow C'(y) = 4y \Rightarrow$$

$$\Rightarrow C(y) = \int 4y dy = \underline{\underline{2y^2 + C}} \quad C \in \mathbb{R}$$

Výsledok: $\varphi(x,y) = \frac{1}{2}x^2y^4 + x^3 + 2y^2 + C$

$$C \in \mathbb{R}$$

Najděte hodnotu parametru $a \in \mathbb{R}$ takovou, aby
vektorové pole $\vec{F} = (4x^3y^4 + axy)\vec{i} + (4x^4y^3 + \frac{1}{2}x^2)\vec{j}$
bylo gradientem nějaké skalární funkce a této funkce.

Najděte.

Označme: $M = 4x^3y^4 + axy$

$N = 4x^4y^3 + \frac{1}{2}x^2$

Podmínka: $\frac{\partial M}{\partial y} = 16x^3y^3 + ax$

$\frac{\partial N}{\partial x} = 16x^3y^3 + x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow 16x^3y^3 + ax = 16x^3y^3 + x \Rightarrow ax = x \Rightarrow a = 1$

\vec{F} je gradientem pro $a = 1$.

$\varphi(x, y) = \int M dx = \int (4x^3y^4 + axy) dx = x^4y^4 + \frac{1}{2}x^2y + C(y)$

$\frac{\partial \varphi}{\partial y} = 4x^4y^3 + \frac{1}{2}x^2 + C'(y)$

$N = 4x^4y^3 + \frac{1}{2}x^2$

$\Downarrow \quad \Leftarrow$
 $\frac{\partial \varphi}{\partial y} = N \Rightarrow C'(y) = 0$

$C(y) = \int 0 dy = C$

Výsledek: $\varphi(x, y) = x^4y^4 + \frac{1}{2}x^2y + C$

$C \in \mathbb{R}$

PŘÍKLAD

Pro funkci $f(x,y) = x^2 + \frac{x}{y^2}$ napište

- a) gradient a gradient v bodě (2,1)
- b) lineární aproximaci v okolí bodu (2,1)
- c) rovnici tečny v bodě (2,1)
- d) totální diferenciál
- e) rovnici tečny k úvratnicové ploše bodem (2,1)

$$\frac{\partial f}{\partial x} = 2x + \frac{1}{y^2}$$

$$\left. \frac{\partial f}{\partial x} \right|_{\substack{x=2 \\ y=1}} = 2 \cdot 2 + \frac{1}{1} = 5$$

$$\frac{\partial f}{\partial y} = -2 \frac{x}{y^3}$$

$$\left. \frac{\partial f}{\partial y} \right|_{\substack{x=2 \\ y=1}} = -2 \frac{2}{1} = -4$$

$$\text{ada) } \nabla f(x,y) = \left(2x + \frac{1}{y^2}, -\frac{2x}{y^3} \right)$$

$$\nabla f(2,1) = (5, -4)$$

$$\text{adb) } f(x,y) \approx 6 + 5(x-2) - 4(y-1) \quad \text{protože } f(2,1) = 6$$

$$\text{adc) } z = 6 + 5(x-2) - 4(y-1) = 5x - 4y$$

$$\text{add) } df = \left(2x + \frac{1}{y^2} \right) dx - 2 \frac{x}{y^3} dy$$

$$\text{ade) } 0 = (5, -4) \cdot (x-2, y-1)$$

$$0 = 5(x-2) - 4(y-1)$$

$$0 = 5x - 4y - 6$$

PRÍKLAD

Určite, pre ktoré $a \in \mathbb{R}$ je výraz

①

$$xy^4 dx + (ax^2y^3 + ay) dy \quad (*)$$

totálnym diferenciálom. Určte konštantu funkcie.

$$M = xy^4 \quad \frac{\partial M}{\partial y} = 4xy^3$$

$$N = ax^2y^3 + ay \quad \frac{\partial N}{\partial x} = 2axy^3$$

Podmienka na totálny diferenciál: $4xy^3 = 2axy^3$
 $\underline{\underline{a = 2}}$

Výraz (*) je totálnym diferenciálom pre $a = 2$.

Hľadáme konštantu funkcie:

$$\begin{aligned} a) \quad \frac{\partial f}{\partial x} = M = xy^4 &\Rightarrow f(x,y) = \int xy^4 dx + C(y) = \\ &= \underline{\underline{\frac{1}{2}x^2y^4}} + \underline{\underline{C(y)}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= \frac{1}{2}x^2 \cdot 4y^3 + C'(y) \\ \frac{\partial f}{\partial y} &= N = \cancel{xy^4} 2x^2y^3 + 2y \end{aligned} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \frac{1}{2}x^2 \cdot 4y^3 + C'(y) = 2x^2y^3 + 2y \\ 2x^2y^3 + C'(y) = 2x^2y^3 + 2y \\ C'(y) = 2y \end{array}$$

$$C(y) = \int C'(y) dy = \int 2y dy = \underline{\underline{y^2}} + C \quad C \in \mathbb{R}$$

Konštantu funkcie je $f(x,y) = \frac{1}{2}x^2y^4 + y^2 + C$
pre ľubovoľnú $C \in \mathbb{R}$.

PRÍKLAD

Výhľad M druhej vrátnice z hmotnosti.

dréna m a hmotnosť. súvisí Mod horcem

$$M = \frac{\mu - M_{od}}{M_{od}}$$

Naměřeno : $\mu = 3,6 \pm 0,2$; $M_{od} = 2,2 \pm 0,1$
kol. 2 je M ?

$$\frac{\partial M}{\partial \mu} = \frac{1}{M_{od}} \quad \left. \frac{\partial M}{\partial \mu} \right|_{\substack{\mu=3,6 \\ M_{od}=2,2}} = \frac{1}{2,2} \approx 0,4545$$

$$\frac{\partial M}{\partial M_{od}} = -\frac{\mu}{M_{od}^2} \quad \left. \frac{\partial M}{\partial M_{od}} \right|_{\substack{\mu=3,6 \\ M_{od}=2,2}} = -\frac{3,6}{2,2^2} = -0,74$$

$$\left(\begin{array}{cc} \frac{\partial M}{\partial \mu} \cdot \Delta \mu & \frac{\partial M}{\partial M_{od}} \cdot \Delta M_{od} \end{array} \right) = \left(\begin{array}{cc} 0,09090 & -0,074 \end{array} \right)$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ & 0,2 & & 0,1 \\ 0,4545 & & -0,74 & \end{array}$$

$$\Delta M = \sqrt{0,09090^2 + 0,074^2} = 0,117 \approx 0,2$$

$$M = \frac{3,6 - 2,2}{2,2} = 0,6363 \approx 0,6$$

$$M = 0,6 \pm 0,2$$

VARIANTA 2 : $\Delta M = |0,09090| + |0,074| = 0,165 \geq 0,2$