1 $\boxed{y' = \frac{y}{x^3}} = x^{-3} \cdot y$

$$\int x^{-3}dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$\underline{\underline{y = c \cdot e^{-\frac{1}{2x^2}}}} \quad | \quad c \in \mathbb{R}$$

$$y' + a(x)\, y = 0$$

$$y' = -a(x)\, y \Rightarrow y = c \cdot e^{-\int a(x)dx}$$

2

$$\boxed{y' - x^2 y = 0}$$

$$y' = \underbrace{x^2}\, y$$

$$\int x^2 dx = \frac{x^3}{3}$$

$$y = c \cdot e^{\frac{x^3}{3}}$$

$$\int x^2 dx = \frac{x^3}{3} + c_1$$

$$y = c \cdot e^{\frac{x^3}{3} + c_1}$$

$$= \underbrace{c} \cdot e^{\frac{x^3}{3}} \cdot \underbrace{e^{c_1}}$$

$$= C_2 \cdot e^{\frac{x^3}{3}}, \quad C_2 = c \cdot e^{c_1}$$

$$C_2 \in \mathbb{R}$$

(3) $\boxed{y' = \dfrac{xy}{x^2+1} \quad , \quad y(0) = 2}$

$$y' = \frac{x}{x^2+1} \cdot y$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$$

$$y = c \cdot e^{\frac{1}{2}\ln(x^2+1)} = c \cdot e^{\ln(x^2+1)^{\frac{1}{2}}}$$

$$y = c \cdot (x^2+1)^{1/2}$$

$$\underline{\underline{y = c \cdot \sqrt{x^2+1}}} \quad , \quad c \in \mathbb{R}$$

$y(0) = 2$:

$$2 = c \cdot \sqrt{0^2+1}$$

$$\underline{\underline{2 = c}}$$

$$\Downarrow$$

$$\boxed{y_p = 2\sqrt{x^2+1}}$$

řešení počáteční úlohy

④ $\boxed{y' + 2y = e^{3x}}$

$$y' + a(x)\,y = b(x)$$

integrační faktor $e^{\int a(x)\,dx}$

$$\int 2\,dx = 2x \Rightarrow \underline{\underline{e^{2x}}}$$

$$\underbrace{y' \cdot e^{2x} + 2y \cdot e^{2x}} = e^{3x} \cdot e^{2x}$$

$$\left(y \cdot e^{2x}\right)' = e^{5x}$$

$$\int e^{5x}\,dx = \frac{1}{5} e^{5x}$$

$$y \cdot e^{2x} = \frac{1}{5} e^{5x} + c \quad / \cdot e^{-2x}$$

$$e^{-2x} = \frac{1}{e^{2x}}$$

$$y = e^{-2x}\left(\frac{1}{5} e^{5x} + c\right)$$

$$\boxed{y = \frac{1}{5} e^{3x} + c \cdot e^{-2x}}$$

(5) $\boxed{y' + y + x^2 \cdot e^{-x} = 0}$

$$y' + y = -x^2 \cdot e^{-x} \quad / \cdot e^{x}$$

integrační faktor je e^{x}

$\int 1 \, dx = x$

$$\underbrace{y' \cdot e^{x} + y \cdot e^{x}}_{(y \cdot e^{x})'} = -x^2 \cdot e^{-x} \cdot e^{x}$$

$$(y \cdot e^{x})' = -x^2$$

$$\int -x^2 \, dx = -\frac{x^3}{3} + c$$

$$y \cdot e^{x} = -\frac{x^3}{3} + c \quad / \cdot e^{-x}$$

$$\underline{\underline{y = e^{-x}\left(c - \frac{x^3}{3}\right)}}, \quad c \in \mathbb{R}$$

(6) $\boxed{y' - \frac{1}{x} y = x \, , \quad y(1) = -1}$

$\int -\frac{1}{x} dx = -\ln x \Rightarrow$ integrační faktor $e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

$y' \cdot \frac{1}{x} - \frac{1}{x} y \cdot \frac{1}{x} = x \cdot \frac{1}{x}$

$\underbrace{y' \cdot \frac{1}{x} - \frac{1}{x^2} y} = 1$

$\left(y \cdot \frac{1}{x} \right)' = 1$

$y \cdot \frac{1}{x} = x + c$

$y = x^2 + cx \, , \quad c \in \mathbb{R}$

$y(1) = -1:$

$-1 = 1^2 + c \cdot 1$

$c = -2$

$\Downarrow$

$\boxed{y_p = x^2 - 2x}$

7 $\boxed{y' - \frac{y}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{x^2}}$

$$y' - \frac{1}{2\sqrt{x}} \cdot y = \frac{e^{\sqrt{x}}}{x^2}$$

$$\underbrace{y' \cdot e^{-\sqrt{x}} - \frac{1}{2\sqrt{x}} y \cdot e^{-\sqrt{x}}} = \frac{e^{\sqrt{x}}}{x^2} \cdot e^{-\sqrt{x}}$$

$$\left(y \cdot e^{-\sqrt{x}}\right)' = \frac{1}{x^2}$$

$$y \cdot e^{-\sqrt{x}} = -\frac{1}{x} + c \quad \Big/ \cdot e^{\sqrt{x}}$$

$$\underline{\underline{y = e^{\sqrt{x}}\left(c - \frac{1}{x}\right)}} \quad , \quad c \in \mathbb{R}$$

$$\int -\frac{1}{2\sqrt{x}}\, dx = -\frac{1}{2} \int \frac{1}{x^{1/2}}\, dx = -\frac{1}{2} \int x^{-\frac{1}{2}}\, dx$$

$$= -\frac{1}{2} \cdot \frac{x^{1/2}}{1/2} = -x^{1/2} = \underline{\underline{-\sqrt{x}}}$$

$\Rightarrow$ integrační faktor $\underline{\underline{e^{-\sqrt{x}}}}$

$$\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + c$$

Příklad 5 ještě jednou – ukázka VARIACE KONSTANTY

$$y' + y + x^2 \cdot e^{-x} = 0$$

y_h: $y' + y = 0$

$$y' = -y$$

$$y_h = c \cdot e^{-x}$$

$$y_p = k(x) \cdot e^{-x}$$

$$y_p' = k'(x) \cdot e^{-x} + k(x) \cdot e^{-x} \cdot (-1)$$

$$y = y_h + y_p$$

y_p, y_p' – dosadíme do rovnice:

$$\underbrace{k'(x) \cdot e^{-x} + k(x) \cdot e^{-x} \cdot (-1)}_{y_p'} + \underbrace{k(x) \cdot e^{-x}}_{y_p} + x^2 \cdot e^{-x} = 0$$

$$k'(x) \cdot e^{-x} + x^2 \cdot e^{-x} = 0$$

$$k'(x) = -x^2 \Rightarrow k(x) = -\frac{x^3}{3}$$

$$y_p = -\frac{x^3}{3} \cdot e^{-x}$$

$$\boxed{y = c \cdot e^{-x} - \frac{x^3}{3} \cdot e^{-x} = e^{-x}\left(c - \frac{x^3}{3}\right)}$$