1. Najděte vlastní hodnoty a vektory matice $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$$A \cdot \vec{u} = \lambda \cdot \vec{u}$$

$$(A - \lambda \cdot I) \cdot \vec{u} = \vec{o}$$

$$\underbrace{\begin{pmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix}}_{\det = 0} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)\cdot(-\lambda) - 2\cdot(-1)$$

$$= -3\lambda + \lambda^2 + 2 = (\lambda - 1)(\lambda - 2)$$

$$\Rightarrow \underline{\underline{\lambda_1 = 1}}, \quad \underline{\underline{\lambda_2 = 2}}$$

- $\underline{\underline{\lambda_1 = 1}}$: $\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2u_1 - u_2 = 0$$

$$u_1 = t, \; u_2 = 2t \quad \Rightarrow u = \begin{pmatrix} t \\ 2t \end{pmatrix}, \; t \in \mathbb{R}$$

volíme $t = 1$: $\boxed{u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

- $\underline{\underline{\lambda_2 = 2}}$: $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$u_1 - u_2 = 0 \quad \Rightarrow \boxed{u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$u_2 = u_1$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \end{array}\right)$$

② Vlastní hodnoty a vektory matice $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\begin{pmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (-2-\lambda)\cdot(1-\lambda) - 4$$

$$= \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$$

$$\Rightarrow \underline{\underline{\lambda_1 = 2}}, \quad \underline{\underline{\lambda_2 = -3}}$$

• $\underline{\underline{\lambda_1 = 2}}$: $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2u_1 - u_2 = 0$$

$$u_1 = t, \; u_2 = 2t \Rightarrow \underline{\underline{u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$$

• $\underline{\underline{\lambda_2 = -3}}$: vektor $\underline{\underline{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}}$

(vektory jsou kolmé, protože matice A je symetrická)

skalární součin: $-2\cdot 1 + 1\cdot 2 = \underline{\underline{0}}$

③ Vlastní hodnoty a vektory matice $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$\underline{\underline{\lambda_1 = 2}}$, $\underline{\underline{\lambda_2 = 3}}$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) = 0$$

$\underline{\underline{\lambda_1 = 2}}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$2u_1 = 2u_1 \implies u_1 = 1 \implies \underline{\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}$$

$$3u_2 = 2u_2 \implies u_2 = 0$$

$\underline{\underline{\lambda_2 = 3}}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underline{3} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$2u_1 = 3u_1 \implies u_1 = 0$$

$$3u_2 = 3u_2 \implies u_2 = 1$$

$$\implies \underline{\underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

④ Vlastní hodnoty a vektory matice $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

vlastní hodnota $\underline{\underline{\lambda = 2}}$

$$\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

<u>každý vektor</u> je vlastním vektorem

⑤ Vlastní hodnoty a vektory matice $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix}$.

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{vmatrix}$$

$$\begin{array}{ccc} \hdashline 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \end{array}$$

$$= (1-\lambda)^2 \cdot (-\lambda) + \underbrace{4 - 8 + 4\lambda}_{-4(1-\lambda)} - 4(1-\lambda) + 2(1-\lambda)$$

$$= (1-\lambda)\left[-\lambda(1-\lambda) - 6\right]$$

$$= (1-\lambda)(\lambda^2 - \lambda - 6)$$

$$= (1-\lambda)(\lambda+2)(\lambda-3) = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = -2, \quad \lambda_3 = 3$$

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = -2,\ \lambda_3 = 3 \qquad \begin{pmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• $\underline{\underline{\lambda_1 = 1}}$:

$$\left(\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ -1 & -1 & 2 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right) \begin{array}{l} /\cdot(-2) \\ \\ + \end{array} \sim \left(\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ \cancel{0} & \cancel{4} & \cancel{-4} & \cancel{0} \end{array}\right)$$

$$2u_2 - 2u_3 = 0, \quad u_3 = t$$

$$u_2 = u_3 = t$$

$$-u_1 - t + 2t = 0$$

$$u_1 = t$$

$$\Rightarrow u = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}}$$

$$\begin{pmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $\underline{\lambda_1 = -2}$: $\left(\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ -1 & 2 & 2 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 3 & 2 & -2 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right) \begin{array}{l} \mid \cdot 3 \mid \cdot(-2) \\ + \\ + \end{array} \sim \left(\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ \cancel{0} & \cancel{8} & \cancel{4} & \cancel{0} \\ 0 & -2 & -1 & 0 \end{array}\right)$

$$-2u_2 - u_3 = 0$$

$\underline{\underline{u_2 = t}}$, $\underline{\underline{u_3 = -2t}}$, $-u_1 + 2t - 4t = 0$

$\underline{\underline{u_1 = -2t}}$

$$t = 1: \underline{\underline{\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}}}$$

$$\begin{pmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• $\underline{\lambda_3 = 3}$:

$$\left(\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ -1 & -3 & 2 & 0 \\ \cancel{-2} & \cancel{2} & \cancel{-2} & \cancel{0} \end{array}\right) /:2 \sim \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -1 & -3 & 2 & 0 \end{array}\right) \begin{array}{l} /\cdot(-1) \\ + \end{array} \sim \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -4 & 3 & 0 \end{array}\right)$$

$$-4u_2 + 3u_3 = 0$$

$$\underline{\underline{u_2 = 3}},\ \underline{\underline{u_3 = 4}} \Rightarrow -u_1 + 3 - 4 = 0$$

$$\underline{\underline{u_1 = -1}}$$

$$\underline{\underline{u = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}}} \Leftarrow$$