1) $y'' - 5y' + 6y = 0$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda_1 = 2, \ \lambda_2 = 3 \Rightarrow y_1 = e^{2x}, \ y_2 = e^{3x}$$

$$y = c_1 \cdot e^{2x} + c_2 \cdot e^{3x}, \quad c_1, c_2 \in \mathbb{R}$$

(2) $y'' + 4y = 0$

$$\lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\lambda = \pm 2i$$

$$\left.\begin{aligned} y_1 &= e^{0x} \cdot \cos 2x = \cos 2x \\ y_1 &= \sin 2x \end{aligned}\right\} \Rightarrow y = C_1 \cdot \cos 2x + C_2 \cdot \sin 2x$$

3) $y'' + 2y' + y = 0 \quad , \; y(1) = 0 , \; y'(1) = 1$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_{1,2} = -1$$

$$y_1 = e^{-x} \quad , \; y_2 = x \cdot e^{-x}$$

$$\boxed{y = c_1 \cdot e^{-x} + c_2 x \cdot e^{-x}}$$

obecné řešení

$$y' = c_1 \cdot e^{-x} \cdot (-1) + c_2 \cdot \left(1 \cdot e^{-x} + x \cdot e^{-x} \cdot (-1)\right)$$

$$y' = -c_1 \cdot e^{-x} + c_2 e^{-x} - c_2 x \cdot e^{-x}$$

$$y(1) = 0 : \; 0 = c_1 \cdot e^{-1} + c_2 e^{-1}$$

$$y'(1) = 1 : \; 1 = -c_1 \cdot e^{-1} + c_2 \cdot e^{-1} - c_2 e^{-1}$$

$+$

$$1 = c_2 \cdot e^{-1} \implies c_2 = e$$

$$0 = c_1 \cdot e^{-1} + 1$$

$$c_1 \cdot e^{-1} = -1 \implies c_1 = -e$$

$$\implies \boxed{y_p} = -e \cdot e^{-x} + e \cdot x \cdot e^{-x} = e^{1-x}(x-1)$$

④ $y'' + 2y' + y = e^{-x}$

$f(x) = e^{-x}$

$\Downarrow$

$y = y_h + y_p$

$y_h: \; y'' + 2y' + y = 0$

$\lambda_{1,2} = -1$ – dvojnás. koren

$$\boxed{y_h = C_1 \cdot e^{-x} + C_2 \cdot x \cdot e^{-x}}$$

$y_p = e^{-x} \cdot a \cdot x^2$ ... dosadíme do rovnice:

$y_p = ax^2 \cdot e^{-x}$

$y_p' = 2ax \cdot e^{-x} + ax^2 \cdot e^{-x} \cdot (-1)$

$y_p'' = 2a \cdot e^{-x} + 2ax \cdot e^{-x} \cdot (-1) + 2ax \cdot e^{-x} \cdot (-1) + ax^2 \cdot e^{-x}$

$= 2a \cdot e^{-x} - 4ax \cdot e^{-x} + ax^2 \cdot e^{-x}$

$$\Rightarrow \text{do rovnice: } \underbrace{2a \cdot e^{-x} - \cancel{4ax \cdot e^{-x}} + \cancel{ax^2 \cdot e^{-x}}}_{y_p''} + \underbrace{\cancel{4ax \cdot e^{-x}} - \cancel{2ax^2 e^{-x}}}_{2y_p'} + \underbrace{\cancel{ax^2 e^{-x}}}_{y_p} = e^{-x}$$

$2a \cdot e^{-x} = e^{-x}$

$2a = 1 \Rightarrow a = \frac{1}{2} \Rightarrow \boxed{y_p = \frac{1}{2} x^2 \cdot e^{-x}} \Rightarrow \boxed{y = C_1 e^{-x} + C_2 x \cdot e^{-x} + \frac{1}{2} x^2 e^{-x}}$

(5) $\underline{y'' + y = x^3}$

$f(x) = x^3$ exp. fce: $e^{0x} = e^0 = 1$

0 není kořen charakt. rce

$\Downarrow$

$y_p = ax^3 + bx^2 + cx + d$

y_h: $y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$

$y_p' = 3ax^2 + 2bx + c$

$y_p'' = 6ax + 2b$

dosadíme: $\underbrace{6ax + 2b}_{y_p''} + \underbrace{ax^3 + bx^2 + cx + d}_{y_p} = x^3$

porovnáme koeficienty u jednotlivých mocnin:

x^3: $a = 1$

x^2: $b = 0$

x: $6a + c = 0 \Rightarrow c = -6$

konst: $2b + d = 0 \Rightarrow d = 0$

$y_1 = \cos x$

$y_2 = \sin x$

$$\boxed{y_h = C_1 \cdot \cos x + C_2 \cdot \sin x}$$

$$\Rightarrow \boxed{y_p = x^3 - 6x}$$

$$\boxed{y = C_1 \cdot \cos x + C_2 \cdot \sin x + x^3 - 6x}$$

(6) Najděte obecné řešení rovnice $y'' - 3y' + 2y = 0$ a napište tvar partikulárního řešení (s neurčitými koeficienty) rovnic:

a) $y'' - 3y' + 2y = e^{x}(3-4x)$

b) $y'' - 3y' + 2y = e^{2x}$

c) $y'' - 3y' + 2y = e^{5x}$

d) $y'' - 3y' + 2y = x^2 \cdot e^{5x}$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda_1 = 1, \ \lambda_2 = 2$

$$\underline{\underline{y_h = c_1 \cdot e^{x} + c_2 \cdot e^{2x}}}$$

a) $y_p = e^{x}(ax+b)\cdot x \Rightarrow y_p = e^{x}(ax^2+bx)$

b) $y_p = e^{2x}\cdot a \cdot x \Rightarrow y_p = axe^{2x}$

c) $y_p = e^{5x}\cdot a \Rightarrow y_p = a\cdot e^{5x}$

d) $y_p = e^{5x}\cdot (ax^2+bx+c)$