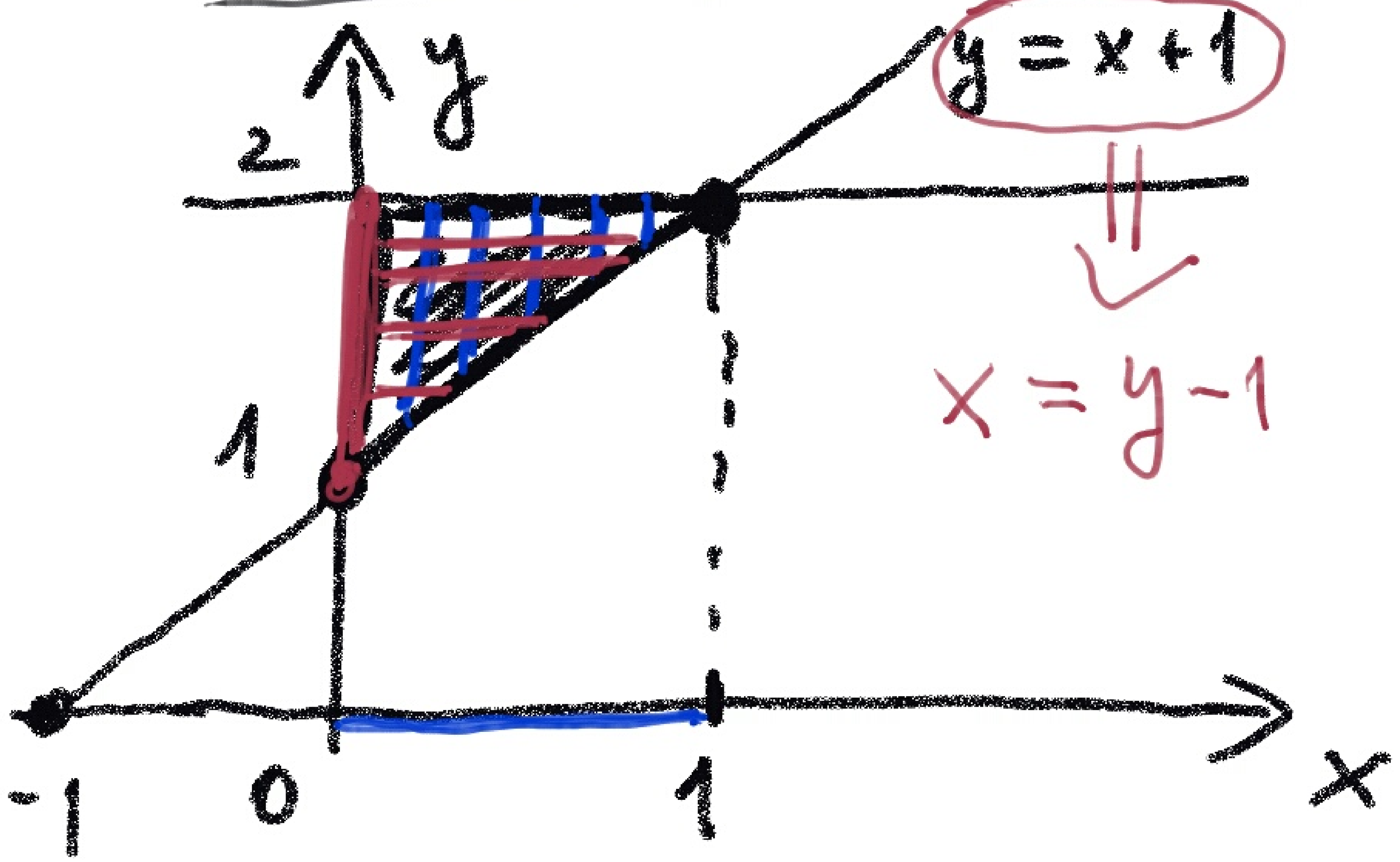
(1) $\iint_M 2xy\,dx\,dy$, M je množina v $\mathbb{R}^2$ ohraničená křivkami

$y=2,\ x=0,\ y=x+1$

Integrál vyjádřete jako dvojnásobný pro obě pořadí integrace a vypočtěte.



$$\int_0^1 \left[\int_{x+1}^{2} 2xy\,dy \right] dx \quad , \quad \int_1^2 \left[\int_0^{y-1} 2xy\,dx \right] dy$$

Výpočet:

$$\int_0^1 \left[\int_{x+1}^2 2xy \, dy \right] dx = \int_0^1 \left[2x \cdot \frac{y^2}{2} \right]_{x+1}^2 dx = \int_0^1 \left[xy^2 \right]_{x+1}^2 dx = \int_0^1 \left[4x - x(x+1)^2 \right] dx$$

$$= \int_0^1 \left(4x - x\left(x^2+2x+1\right)\right) dx = \int_0^1 \left(4x - x^3 - 2x^2 - x\right) dx = \int_0^1 \left(3x - x^3 - 2x^2\right) dx$$

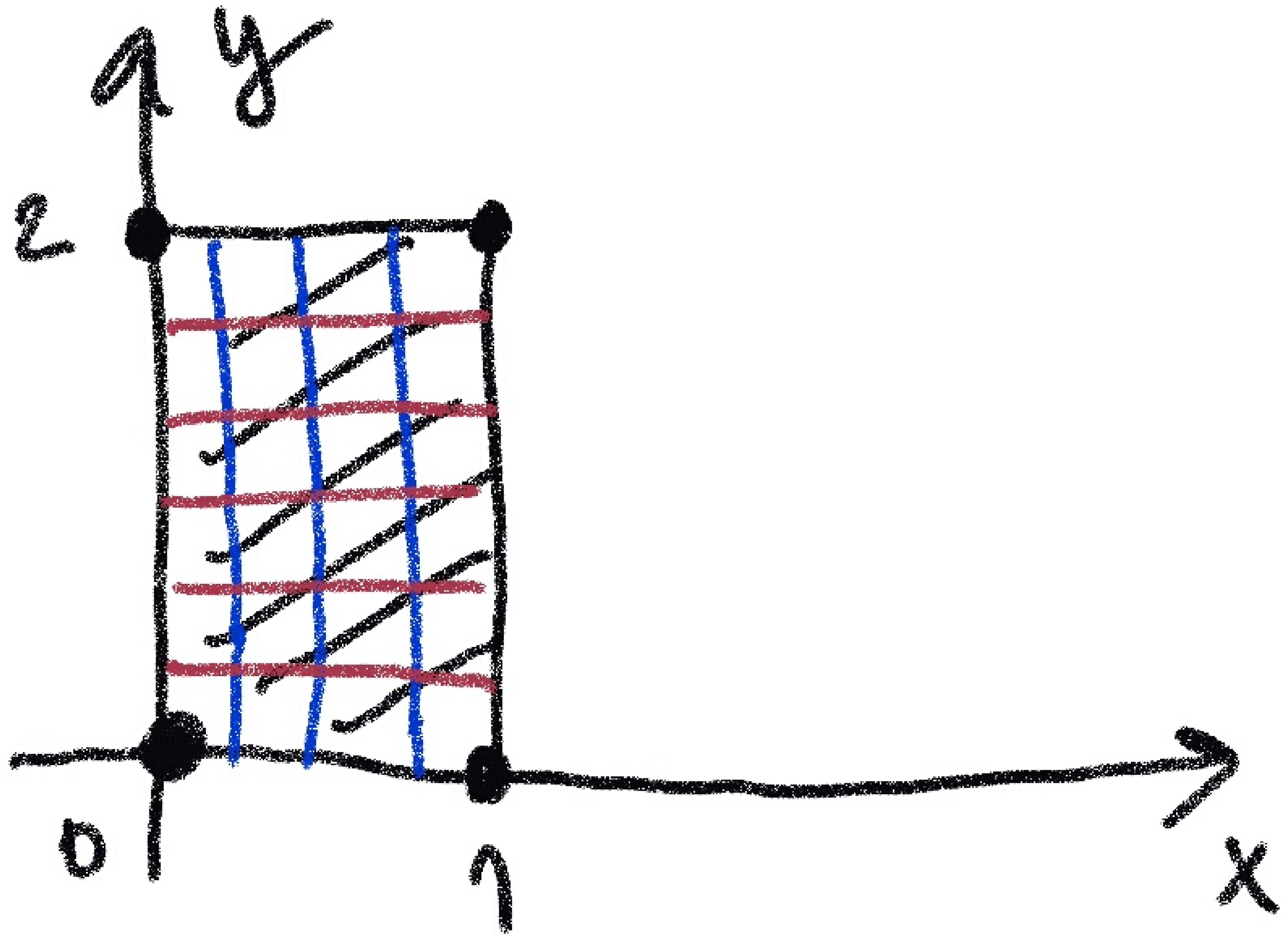
$$= \left[3\frac{x^2}{2} - \frac{x^4}{4} - 2 \cdot \frac{x^3}{3} \right]_0^1 = \frac{3}{2} - \frac{1}{4} - \frac{2}{3} = \frac{18-3-8}{12} = \underline{\underline{\frac{7}{12}}}$$

$$\int_1^2 \left[\int_0^{y-1} 2xy \, dx \right] dy = \int_1^2 \left[2 \cdot \frac{x^2}{2} y \right]_0^{y-1} dy = \int_1^2 \left[x^2 y \right]_0^{y-1} dy = \int_1^2 (y-1)^2 \cdot y \, dy$$

$$= \int_1^2 \left(y^2 - 2y + 1\right) \cdot y \, dy = \int_1^2 \left(y^3 - 2y^2 + y\right) dy = \left[\frac{y^4}{4} - 2 \cdot \frac{y^3}{3} + \frac{y^2}{2} \right]_1^2$$

$$= 4 - \frac{16}{3} + 2 - \frac{1}{4} + \frac{2}{3} - \frac{1}{2} = 6 - \frac{14}{3} - \frac{1}{4} - \frac{1}{2} = \underline{\underline{\frac{7}{12}}}$$

(2) $\iint_M (x^2+2y)\,dx\,dy$, M – obdélník s vrcholy $[0,0]$, $[1,0]$, $[0,2]$, $[1,2]$

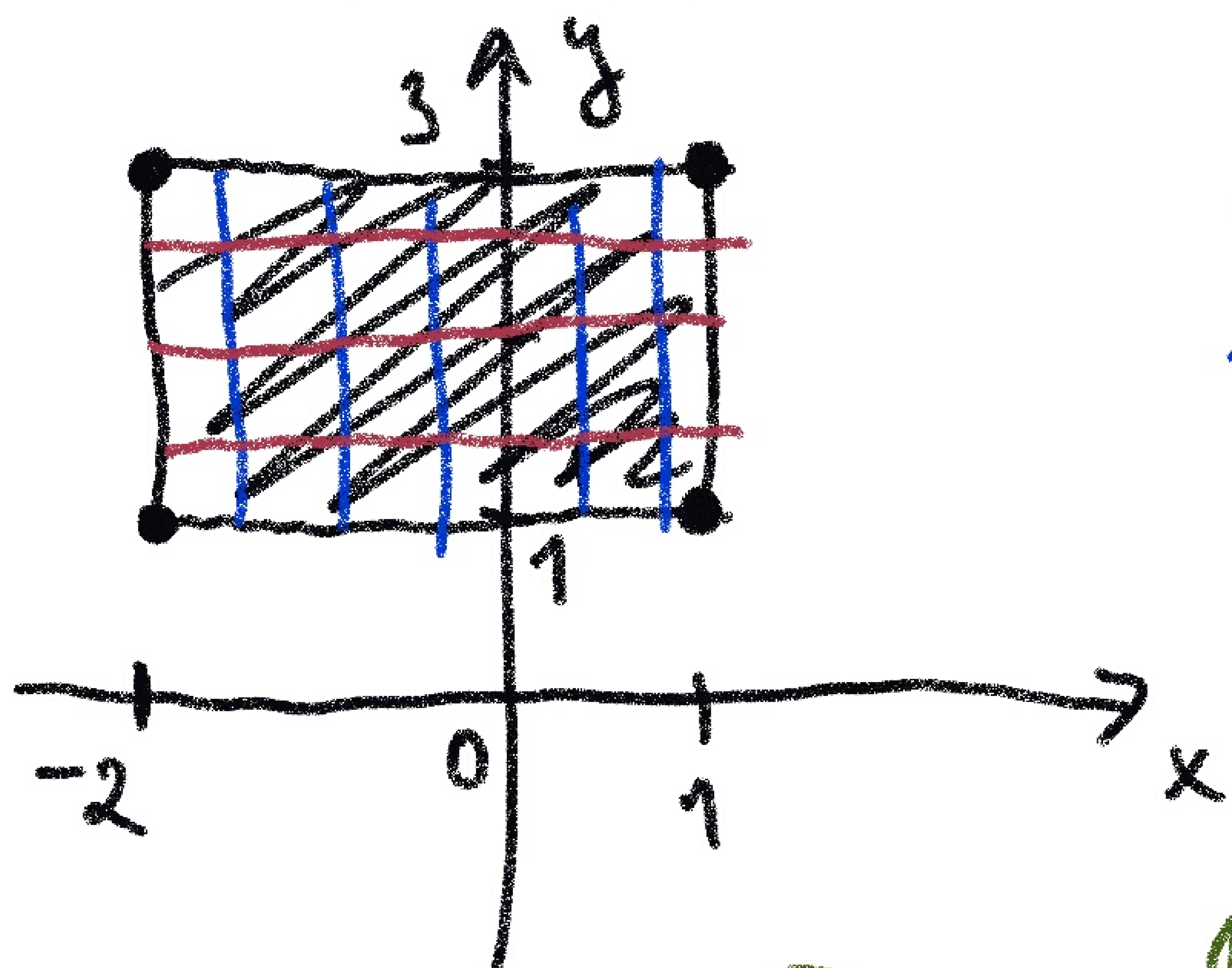


$$\int_0^1 \left[\int_0^2 (x^2+2y)\,dy \right] dx \quad , \quad \int_0^2 \left[\int_0^1 (x^2+2y)\,dx \right] dy$$

$$= \int_0^1 \left[x^2 y + y^2 \right]_0^2 dx = \int_0^1 (2x^2+4)\,dx$$

$$= \left[2\cdot\frac{x^3}{3} + 4x \right]_0^1 = \frac{2}{3} + 4 = \frac{14}{3}$$

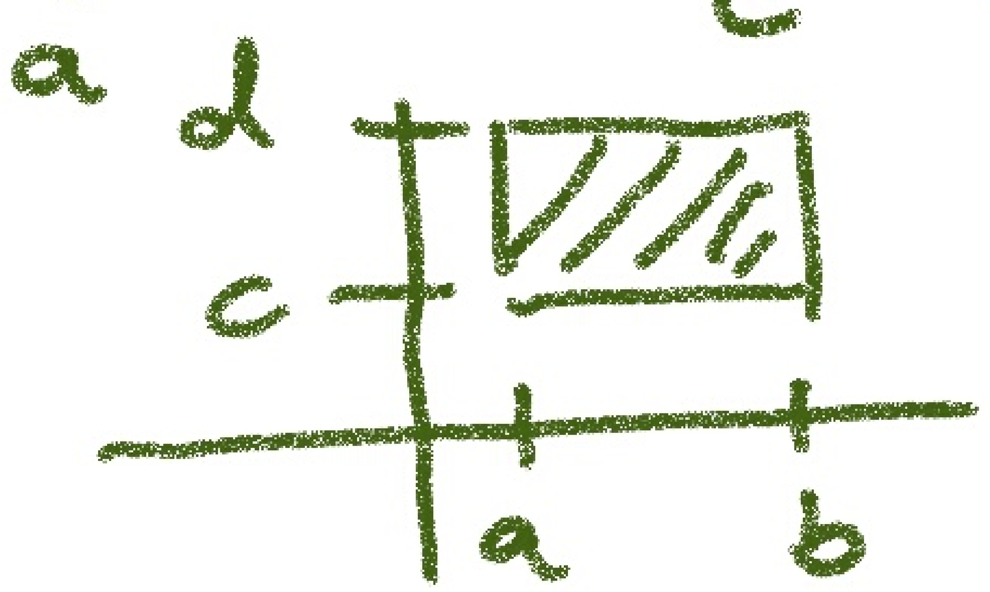
③ $\iint\limits_M \frac{x^2}{y^2}\,dxdy$, M: obdélník s vrcholy $[-2,1]$, $[1,1]$, $[1,3]$, $[-2,3]$



$$\int_{-2}^{1}\left[\int_{1}^{3}\frac{x^2}{y^2}\,dy\right]dx \quad , \quad \int_{1}^{3}\left[\int_{-2}^{1}\frac{x^2}{y^2}\,dx\right]dy$$

$$! \quad \iint\limits_{\square} \underbrace{f(x)}\cdot\underbrace{g(y)}\,dxdy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dx$$

$\square \leftarrow$ obd.

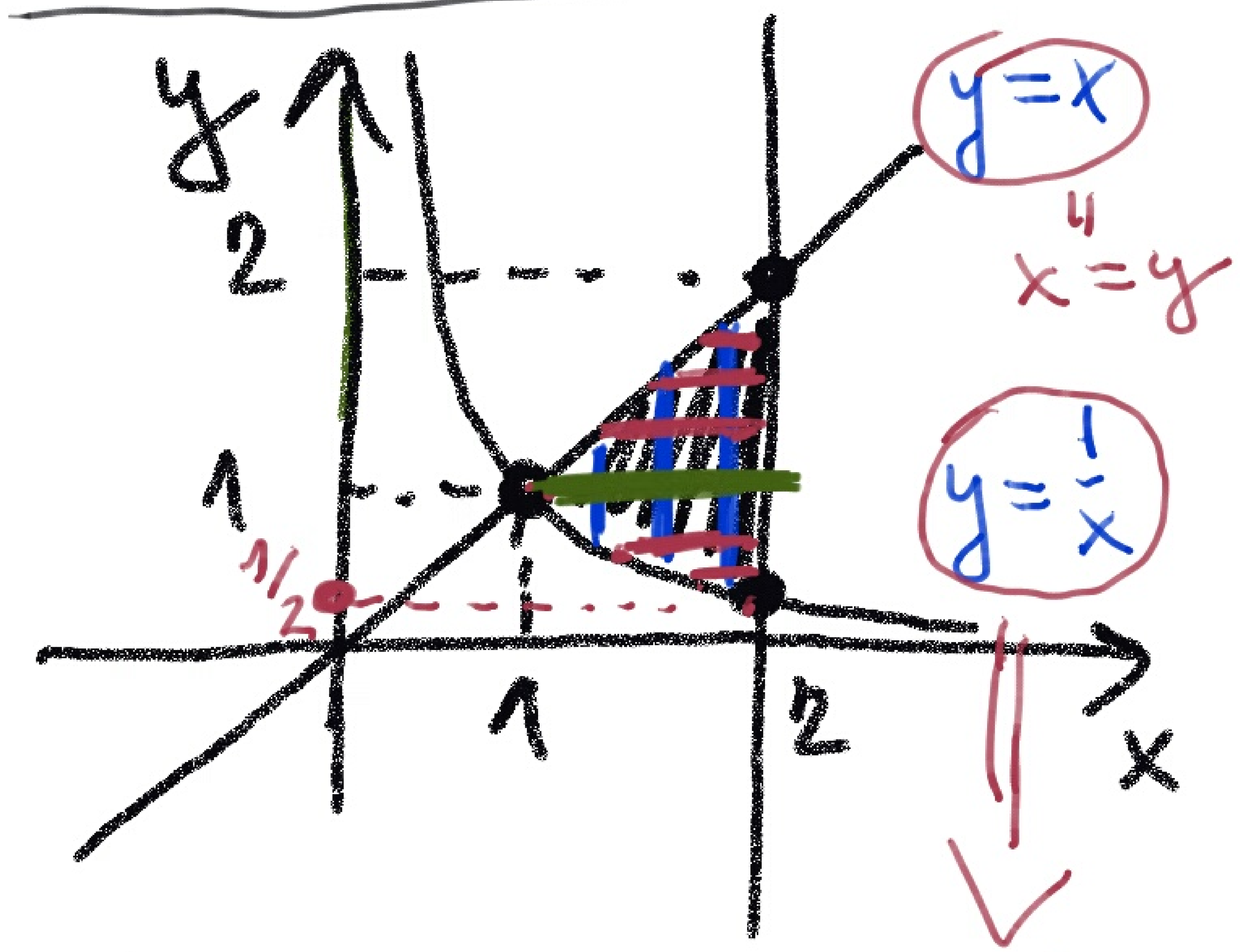


$$\Rightarrow \frac{x^2}{y^2} = x^2 \cdot \frac{1}{y^2} \Rightarrow \int_{-2}^{1} x^2\,dx \cdot \int_1^3 \underbrace{\frac{1}{y^2}}_{=y^{-2}}\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-2}^{1} \cdot \left[\frac{y^{-1}}{-1}\right]_1^3 = \left[\frac{1}{3} - \left(\frac{-8}{3}\right)\right] \cdot \left[-\frac{1}{y}\right]_1^3 = 3 \cdot \left[-\frac{1}{3} + 1\right]$$

$$= 3 \cdot \frac{2}{3} = \boxed{2}$$

④ $\iint\limits_M (2xy+1)\,dx\,dy$, M – ohraničená křivkami $y=\frac{1}{x}$, $y=x$, $x=2$



$\frac{1}{x}=x$ $\quad x=\frac{1}{y}$

$$\int_1^2 \left[\int_{1/x}^{x} (2xy+1)\,dy\right] dx$$

$$\int_{1/2}^{1} \left[\int_{1/y}^{2} (2xy+1)\,dx\right] dy + \int_1^2 \left[\int_y^2 (2xy+1)\,dx\right] dy$$

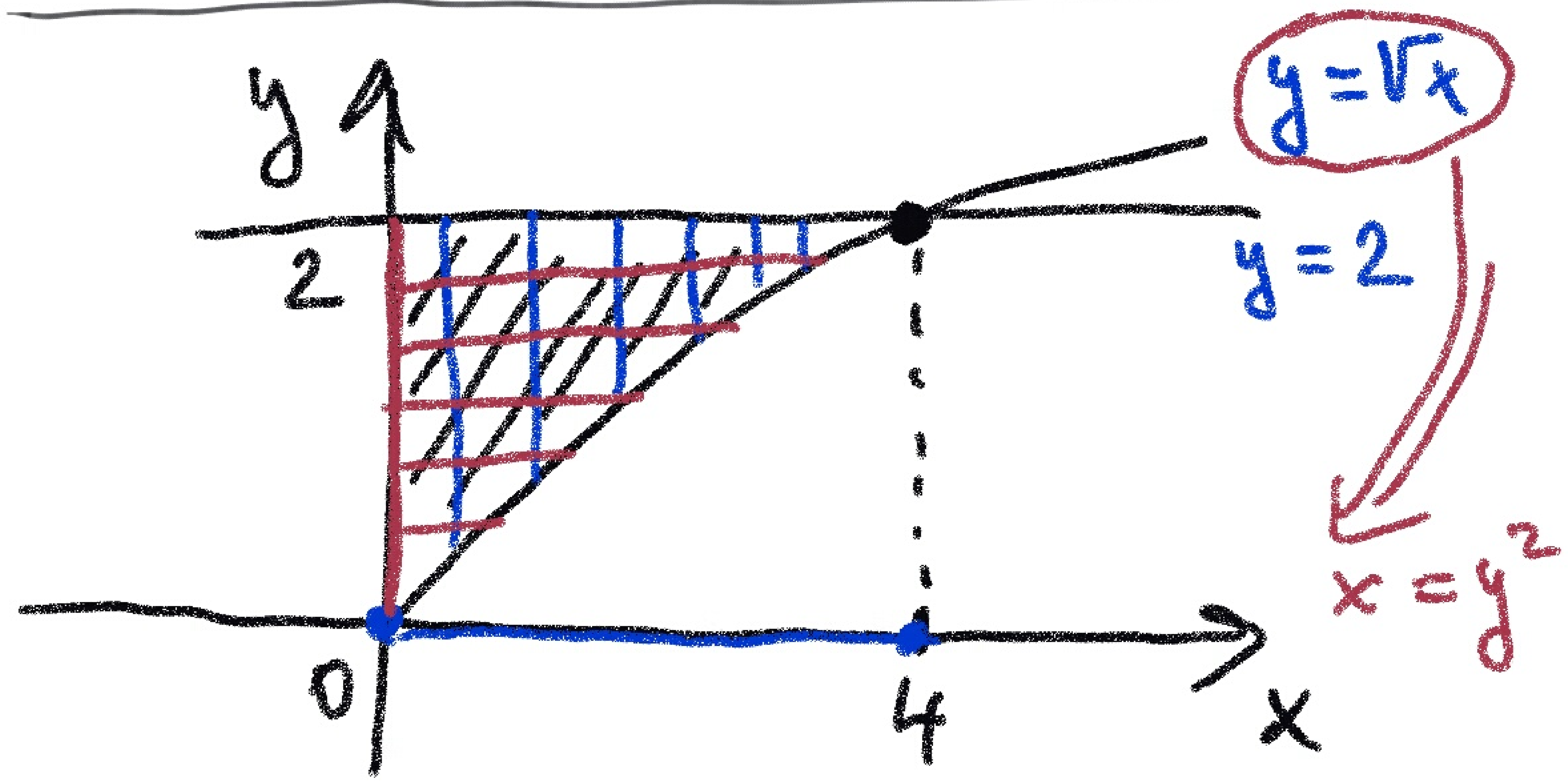
$$\Rightarrow \int_1^2 \left[\int_{1/x}^{x} (2xy+1)\,dy\right] dx = \int_1^2 \left[xy^2+y\right]_{1/x}^{x} dx = \int_1^2 \Big(x^3+x\underbrace{-\frac{1}{x}-\frac{1}{x}}_{-2\cdot\frac{1}{x}}\Big)\,dx$$

$$= \left[\frac{x^4}{4}+\frac{x^2}{2}-2\cdot\ln x\right]_1^2 = 4+2-2\ln 2-\frac{1}{4}-\frac{1}{2}+\underbrace{2\ln 1}_{=0} = 6-2\ln 2-\frac{3}{4}$$

$$= \frac{21}{4}-2\ln 2$$

⑤ Dvojný integrál $\iint\limits_M f(x,y)\,dx\,dy$ vyjádřete jako dvojnásobný pro obě pořadí integrace. M: ohraničená křivkami

$$y=\sqrt{x}\,,\quad x=0\,,\quad y=2$$



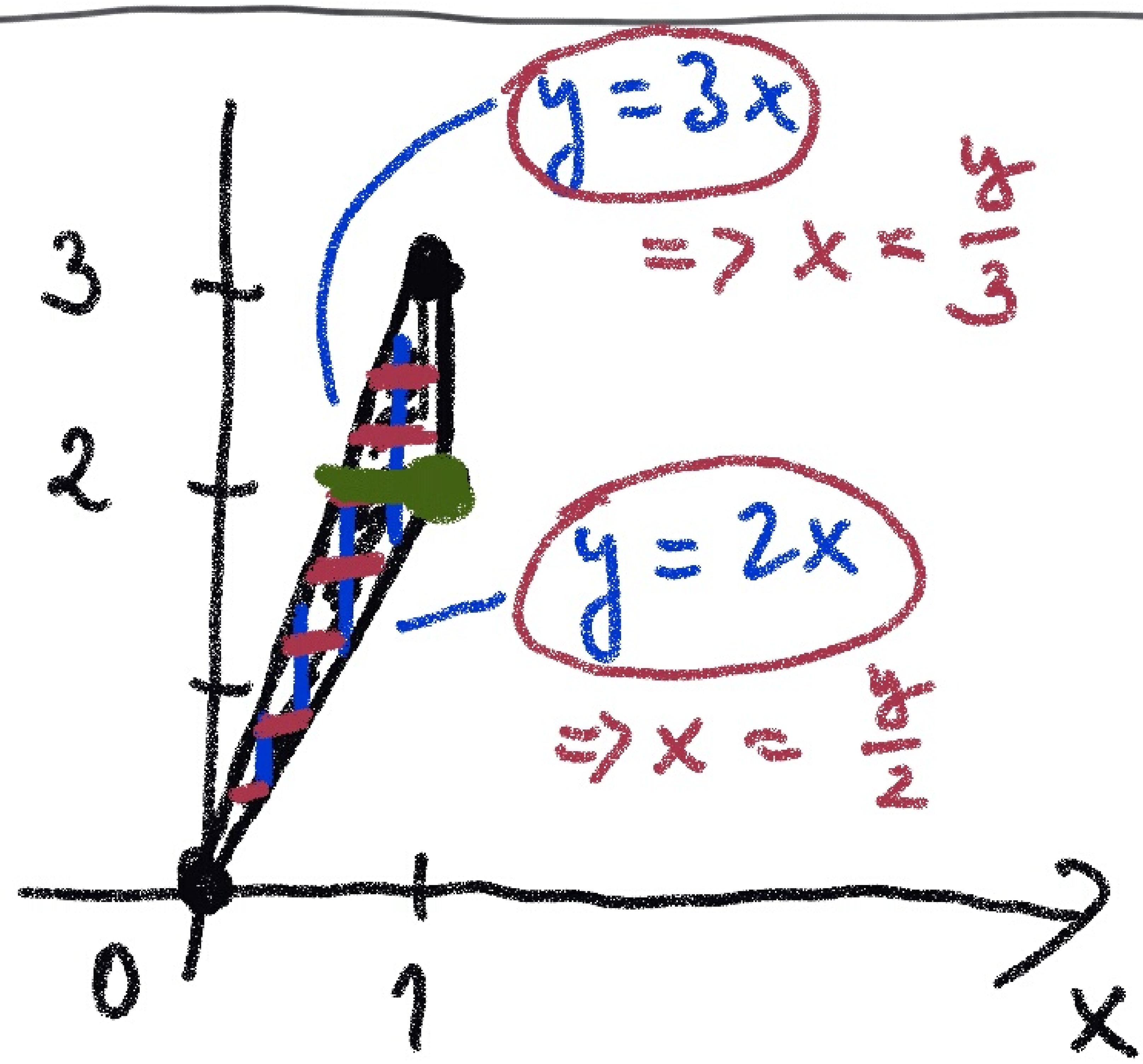
$$\int_0^4 \left[\int_{\sqrt{x}}^2 f(x,y)\,dy\right] dx$$

$$\int_0^2 \left[\int_0^{y^2} f(x,y)\,dx\right] dy$$

$y=\sqrt{x}$

$2=\sqrt{4}$

(6) Dvojný integrál $\iint_M f(x,y)\,dx\,dy$ vyjádřete jako dvojnásobný pro obě pořadí integrace. M: Δ s vrcholy $[0,0]$, $[1,2]$, $[1,3]$



$$\int_0^1 \left[\int_{2x}^{3x} f(x,y)\,dy \right] dx$$

$$\int_0^2 \left[\int_{y/3}^{y/2} f(x,y)\,dx \right] dy + \int_2^3 \left[\int_{y/3}^{1} f(x,y)\,dx \right] dy$$