

# Systems of linear equations

Mathematics – RRMATA

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# Basic concepts

## Definition (System of linear equations)

A **system of  $m$  linear equations in  $n$  unknowns** is a collection of equations

$$\begin{aligned} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ (*) \quad & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{aligned}$$

Variables  $x_1, x_2, \dots, x_n$  are called **unknowns**. Numbers  $a_{ij}$  are called **coefficients of the left-hand sides** and numbers  $b_i$  are called **coefficients of the right-hand sides**.

A **solution** of the system is an ordered  $n$ -tuple of real numbers  $t_1, t_2, \dots, t_n$  that make each equation true statement when the values  $t_1, t_2, \dots, t_n$  are substituted for  $x_1, x_2, \dots, x_n$ , respectively.

If  $b_1 = b_2 = \cdots = b_m = 0$ , the system is called **homogenous**.

## Definition (Coefficient matrix, augmented matrix)

- The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called the **coefficient matrix** of system (\*).

- The matrix

$$\tilde{A} = \left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

is called the **augmented matrix** of system (\*).

## Matrix notation of (\*)

Denote

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the vector of the right-hand sides and unknowns, respectively. System (\*) can be written as the **matrix equation**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

i.e.,

$$A\vec{x} = \vec{b}.$$

## Theorem (Frobenius)

System (\*) has a solution if and only if the rank of the coefficient matrix of (\*) is equal to the rank of the augmented matrix of this system, i.e.,

$$\text{rank}A = \text{rank}\tilde{A}.$$

## Remark

System (\*) may have no solution, exactly one solution, or infinitely many solutions.

- If  $\text{rank}A < \text{rank}\tilde{A}$ , then (\*) has **no solution**.
- If  $\text{rank}A = \text{rank}\tilde{A} = n$ , then (\*) has **exactly one solution**.
- If  $\text{rank}A = \text{rank}\tilde{A} < n$ , then (\*) has **infinitely many solutions**. In this case the unknowns can be computed in terms of  $n - \text{rank}A$  **parameters (free variables)**.

Homogeneous linear systems have either exactly one solution (namely,  $x_1 = 0$ ,  $x_2 = 0$ ,  $\dots$ ,  $x_n = 0$ , called the **trivial solution**) or an infinite number of solutions (including the trivial solution).

## Gauss method

- 1 We convert the augmented matrix  $\tilde{A}$  into its row echelon form (using row operations). We find  $\text{rank}\tilde{A}$  and  $\text{rank}A$  to determine the solvability or nonsolvability of  $(*)$  (Frobenius theorem).
- 2 If  $\text{rank}A = \text{rank}\tilde{A}$ , we rewrite back the row echelon form of  $\tilde{A}$  into a system of linear equations (in the original unknowns). This system has the same set of solutions as the original system  $(*)$ .
- 3 We solve this new system from below:
  - If  $\text{rank}A = \text{rank}\tilde{A} = n$ , there is exactly one “new” unknown in each equation of the system. (Other unknowns have been computed from the equations below.)  
 $\Rightarrow$  **exactly one solution**
  - If  $\text{rank}A = \text{rank}\tilde{A} < n$ , then there exists at least one equation with  $k > 1$  “new” unknowns. In this case, we solve one arbitrary of these unknowns through the other  $k - 1$  unknowns. These  $k - 1$  unknowns are called **free variables** and can be considered as parameters, i.e., they can take any real values  $\Rightarrow$  **infinitely many solutions**. The choice of the free unknowns is not unique, hence the set of solutions can be written in different forms.

### Example (One solution)

Solve the system:

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 0 \\2x_1 + 4x_2 + 7x_3 &= 8 \\3x_1 + 5x_2 + 10x_3 &= 10\end{aligned}$$

$$\left( \begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 2 & 4 & 7 & 8 \\ 3 & 5 & 10 & 10 \end{array} \right) \begin{array}{l} \left[ \begin{array}{l} \leftarrow -2 \\ \leftarrow + \end{array} \right]^{-3} \\ \left[ \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array} \right]^{-3} \end{array} \sim \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & \boxed{2} & 3 & 8 \\ 0 & 2 & 4 & 10 \end{array} \right) \begin{array}{l} \left[ \begin{array}{l} \leftarrow -1 \\ \leftarrow + \end{array} \right]^{-1} \\ \left[ \begin{array}{l} \leftarrow -1 \\ \leftarrow + \end{array} \right]^{-1} \end{array} \sim \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Rank of the coefficient matrix (denote  $A$ ) and of the augmented matrix (denote  $\tilde{A}$ ):

$$\text{rank}(A) = \text{rank}(\tilde{A}) = 3$$

number of variables:  $n = 3$

$\Rightarrow$  1 solution

From the last matrix (solved from below):

$$\boxed{x_3 = 2}$$

$$2x_2 + 3 \cdot 2 = 8 \Rightarrow \boxed{x_2 = 1}$$

$$x_1 + 1 + 2 \cdot 2 = 0 \Rightarrow \boxed{x_1 = -5}$$

### Example (Infinitely many solution, 1 parameter)

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 4$$

Solve the system:

$$x_2 - x_3 + x_4 = -3$$

$$x_1 + 3x_2 - 3x_4 = 1$$

$$-7x_2 + 3x_3 + x_4 = -3$$

$$\left( \begin{array}{cccc|c} \boxed{1} & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 1 & 3 & 0 & -3 & 1 \\ 0 & -7 & 3 & 1 & -3 \end{array} \right) \begin{array}{l} \left[ \begin{array}{l} \leftarrow^{-1} \\ \leftarrow^{+} \end{array} \right] \\ \left[ \begin{array}{l} \leftarrow^{-5} \\ \leftarrow^{+} \end{array} \right] \end{array} \sim \left( \begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & \boxed{1} & -1 & 1 & -3 \\ 0 & 5 & -3 & 1 & -3 \\ 0 & -7 & 3 & 1 & -3 \end{array} \right) \begin{array}{l} \left[ \begin{array}{l} \leftarrow^{-5} \\ \leftarrow^{+} \end{array} \right] \end{array} \sim \left( \begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 & 12 \\ 0 & 0 & -4 & 8 & -24 \end{array} \right) \begin{array}{l} | : 2 \\ \end{array} \sim \left( \begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -2 & 6 \end{array} \right)$$

$$\text{rank}(A) = \text{rank}(\tilde{A}) = 3$$

number of variables:  $n = 4$

$\Rightarrow \infty$  solutions, 1

parameter

$$x_3 - 2x_4 = 6 : \boxed{x_4 = t, t \in \mathbb{R}} \Rightarrow \boxed{x_3 = 6 + 2t}$$

$$x_2 - (6 + 2t) + t = -3 \Rightarrow \boxed{x_2 = 3 + t}$$

$$x_1 - 2(3 + t) + 3(6 + 2t) - 4t = 4 \Rightarrow \boxed{x_1 = -8}$$



### Example (Infinitely many solutions, 2 parameters)

Solve the system:

$$\begin{aligned}x_1 + 2x_2 + 4x_3 - 3x_4 &= 0 \\3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0 \\4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0 \\3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0\end{aligned}$$

$$\left( \begin{array}{cccc|c} \boxed{1} & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & 0 \\ 4 & 5 & -2 & 3 & 0 \\ 3 & 8 & 24 & -19 & 0 \end{array} \right) \begin{array}{l} \left[ \begin{array}{l} \leftarrow -3 \\ \leftarrow -4 \end{array} \right] \left[ \begin{array}{l} \leftarrow -3 \\ \leftarrow -4 \end{array} \right] \\ \left[ \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array} \right] \left[ \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array} \right] \end{array}$$
$$\sim \left( \begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -3 & -18 & 15 & 0 \\ 0 & 2 & 12 & -10 & 0 \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -1 & -6 & 5 & 0 \end{array} \right)$$

$\text{rank}(A) = \text{rank}(\tilde{A}) = 2$   
number of variables:  $n = 4$   
 $\Rightarrow \infty$  solutions, 2  
parameters

$$-x_2 - 6x_3 + 5x_4 = 0 : \boxed{x_4 = t, x_3 = s, t, s \in \mathbb{R}}$$

$$\Rightarrow \boxed{x_2 = -6s + 5t}$$

$$x_1 + 2(-6s + 5t) + 4s - 3t = 0 \Rightarrow \boxed{x_1 = 8s - 7t}$$

### Example (No solution)

Solve the system:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 1 \\2x_1 + x_2 + 2x_3 &= 1 \\4x_1 + 5x_2 + 8x_3 &= 2\end{aligned}$$

$$\begin{aligned}&\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 5 & 8 & 2 \end{array}\right) \begin{array}{l} \leftarrow^{-2} \\ \leftarrow^{+} \\ \leftarrow^{+} \end{array} \begin{array}{l} -4 \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{-3} & -4 & -1 \\ 0 & -3 & -4 & -2 \end{array}\right) \begin{array}{l} \\ \leftarrow^{-1} \\ \leftarrow^{+} \end{array} \\ &\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)\end{aligned}$$

$$\text{rank}(A) = 2, \quad \text{rank}(\tilde{A}) = 3$$

$\text{rank}(A) \neq \text{rank}(\tilde{A}) \implies$  the system has no solution.

## Theorem (Properties of regular matrices)

Let  $A$  be an  $n \times n$  square matrix. Then the following statements are equivalent:

- 1  $A$  is invertible, i.e.,  $A^{-1}$  exists.
- 2  $\det A \neq 0$
- 3  $\text{rank} A = n$ .
- 4 The rows (columns) of  $A$  are linearly independent.
- 5 System of linear equations  $A\vec{x} = \vec{b}$  has a unique solution for any vector  $\vec{b}$ .

# Method of matrix inversion

Next we present a method which can be used for solving the system  $A\vec{x} = \vec{b}$  in case when  $A$  is regular.

## Theorem (Method of matrix inversion)

*Let  $A$  be an  $n \times n$  matrix and suppose that  $A$  is invertible. Then system of equations  $A\vec{x} = \vec{b}$  has a unique solution*

$$\vec{x} = A^{-1}\vec{b}.$$

## Example

Solve the system:

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 1 \\2x_1 + x_2 + 3x_3 &= 2 \\x_1 + x_2 + x_3 &= 3\end{aligned}$$

The coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

The vector of the right-hand sides:

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The inverse matrix of  $A$ :

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The vector of solutions:  $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$$\implies \boxed{x_1 = 3, x_2 = 2, x_3 = -2}.$$

# Using the computer algebra systems

Solve the system using Wolfram Alpha (<http://www.wolframalpha.com/>):

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 1 \\2x_1 + x_2 + 3x_3 &= 2 \\x_1 + x_2 + x_3 &= 3\end{aligned}$$

Solution:

```
solve x1+x2+2*x3=1,2x1+x2+3x3=2,x1+x2+x3=3
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