Systems of linear equations

Mathematics – RRMATA

MENDELU

Basic concepts

Definition (System of linear equations)

A **system of m linear equations in n unknowns** is a collection of equations

$$(*)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Variables x_1, x_2, ..., x_n are called **unknowns**. Numbers a_{ij} are called **coefficients of the left-hand sides** and numbers b_i are called **coefficients of the right-hand sides**.
A **solution** of the system is an ordered n-tuple of real numbers $t_1, t_2, \dots t_n$ that make each equation true statement when the values $t_1, t_2, \dots t_n$ are substituted for x_1, x_2, ..., x_n, respectively.
If $b_1 = b_2 = \cdots = b_m = 0$, the system is called **homogenous**.

Definition (Coefficient matrix, augmented matrix)

- The matrix
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$
is called the **coefficient matrix** of system $(*)$.
- The matrix
$$\tilde{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$
is called the **augmented matrix** of system $(*)$.

Matrix notation of $(*)$

Denote
$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
the vector of the right-hand sides and unknowns, respectively. System $(*)$ can be written as the **matrix equation**
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$
i.e.,
$$A\vec{x} = \vec{b}.$$

Theorem (Frobenius)

System $(*)$ *has a solution if and only if the rank of the coefficient matrix of* $(*)$ *is equal to the rank of the augmented matrix of this system, i.e.,*

$$\text{rank}A = \text{rank}\tilde{A}.$$

Remark

System $(*)$ may have no solution, exactly one solution, or infinitely many solutions.

- If $\text{rank}A < \text{rank}\tilde{A}$, then $(*)$ has no solution.
- If $\text{rank}A = \text{rank}\tilde{A} = n$, then $(*)$ has exactly one solution.
- If $\text{rank}A = \text{rank}\tilde{A} < n$, then $(*)$ has infinitely many solutions. In this case the unknowns can be computed in terms of $n - \text{rank}A$ **parameters (free variables)**.

Homogeneous linear systems have either exactly one solution (namely, $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$, called the **trivial solution**) or an infinite number of solutions (including the trivial solution).

Gauss method

1. We convert the augmented matrix $\tilde{A}$ into its row echelon form (using row operations). We find $\text{rank}\tilde{A}$ and $\text{rank}A$ to determine the solvability or nonsolvability of $(*)$(Frobenius theorem).
2. If $\text{rank}A = \text{rank}\tilde{A}$, we rewrite back the row echelon form of $\tilde{A}$ into a system of linear equations (in the original unknowns). This system has the same set of solutions as the original system $(*)$.
3. We solve this new system from below:
   - If $\text{rank}A = \text{rank}\tilde{A}$=n, there is exactly one "new" unknown in each equation of the system. (Other unknowns have been computed from the equations below.) $\Rightarrow$ **exactly one solution**
   - If $\text{rank}A = \text{rank}\tilde{A} < n$, then there exists at least one equation with $k > 1$ "new" unknowns. In this case, we solve one arbitrary of these unknowns through the other $k-1$ unknowns. These $k-1$ unknowns are called **free variables** and can be considered as parameters, i.e., they can take any real values $\Rightarrow$ **infinitely many solutions.** The choice of the free unknowns is not unique, hence the set of solutions can be written in different forms.

Example (One solution)

Solve the system:
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 + 4x_2 + 7x_3 &= 8 \\ 3x_1 + 5x_2 + 10x_3 &= 10 \end{aligned}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 2 & 4 & 7 & 8 \\ 3 & 5 & 10 & 10 \end{array}\right) \begin{matrix} R_2 + (-2)R_1 \\ R_3 + (-3)R_1 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & \boxed{2} & 3 & 8 \\ 0 & 2 & 4 & 10 \end{array}\right) R_3 + (-1)R_2 \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Rank of the coefficient natrix (denote A) and of the augmented matrix (denote $\tilde{A}$):

$$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = 3$$

number of variables: $n = 3$

$\Rightarrow$ 1 solution

From the last matrix (solved from below):

$\boxed{x_3 = 2}$

$2x_2 + 3 \cdot 2 = 8 \Rightarrow \boxed{x_2 = 1}$

$x_1 + 1 + 2 \cdot 2 = 0 \Rightarrow \boxed{x_1 = -5}$

Example (Infinitely many solution, 1 parameter)

Solve the system:
$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\ x_2 - x_3 + x_4 &= -3 \\ x_1 + 3x_2 \qquad - 3x_4 &= 1 \\ -7x_2 + 3x_3 + x_4 &= -3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 1 & 3 & 0 & -3 & 1 \\ 0 & -7 & 3 & 1 & -3 \end{array}\right) R_3 + (-1)R_1 \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & \boxed{1} & -1 & 1 & -3 \\ 0 & 5 & -3 & 1 & -3 \\ 0 & -7 & 3 & 1 & -3 \end{array}\right) \begin{matrix} R_3 + (-5)R_2 \\ R_4 + 7R_2 \end{matrix}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 & 12 \\ 0 & 0 & -4 & 8 & -24 \end{array}\right) \Big| : 2 \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \end{array}\right)$$

$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = 3$
number of variables: $n = 4$
$\Rightarrow \infty$ solutions, 1 parameter

$x_3 - 2x_4 = 6 : \boxed{x_4 = t,\ t \in \mathbb{R}} \Rightarrow \boxed{x_3 = 6 + 2t}$

$x_2 - (6 + 2t) + t = -3 \Rightarrow \boxed{x_2 = 3 + t}$

$x_1 - 2(3 + t) + 3(6 + 2t) - 4t = 4 \Rightarrow \boxed{x_1 = -8}$

Example (Infinitely many solutions, 2 parameters)

Solve the system:
$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0 \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & 0 \\ 4 & 5 & -2 & 3 & 0 \\ 3 & 8 & 24 & -19 & 0 \end{array}\right) \begin{array}{l} \cdot(-3),\ \cdot(-4),\ \cdot(-3) \\ + \\ + \\ + \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -3 & -18 & 15 & 0 \\ 0 & 2 & 12 & -10 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \end{array}\right)$$

$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = 2$
number of variables: $n = 4$
$\Rightarrow \infty$ solutions, 2 parameters

$-x_2 - 6x_3 + 5x_4 = 0$: $\boxed{x_4 = t, x_3 = s,\ t, s \in \mathbb{R}}$

$\Rightarrow \boxed{x_2 = -6s + 5t}$

$x_1 + 2(-6s + 5t) + 4s - 3t = 0 \Rightarrow \boxed{x_1 = 8s - 7t}$

Example (No solution)

Solve the system:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 1 \\ 4x_1 + 5x_2 + 8x_3 &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 5 & 8 & 2 \end{array}\right) \begin{array}{l} \cdot(-2),\ \cdot(-4) \\ + \\ + \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{-3} & -4 & -1 \\ 0 & -3 & -4 & -2 \end{array}\right) \begin{array}{l} \\ \cdot(-1) \\ + \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$\operatorname{rank}(A) = 2, \quad \operatorname{rank}(\tilde{A}) = 3$$

$\operatorname{rank}(A) \neq \operatorname{rank}(\tilde{A}) \Longrightarrow$ the system has no solution.

Systems with regular coefficient matrices

Theorem (Properties of regular matrices)

Let A be an $n \times n$ square matrix. Then the following statements are equivalent:

1. *A is invertible, i.e., A^{-1} exists.*
2. $\det A \neq 0$
3. $\mathrm{rank} A = n$.
4. *The rows (columns) of A are linearly independent.*
5. *System of linear equations $A\vec{x} = \vec{b}$ has a unique solution for any vector $\vec{b}$.*

Method of matrix inversion

Next we present a method which can be used for solving the system $A\vec{x} = \vec{b}$ in case when A is regular.

Theorem (Method of matrix inversion)

Let A be an $n \times n$ matrix and suppose that A is invertible. Then system of equations $A\vec{x} = \vec{b}$ has a unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

Example

Solve the system:
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

The coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

The vector of the right-hand sides:

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The inverse matrix of A:

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The vector of solutions: $$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$$\Longrightarrow \boxed{x_1 = 3,\ x_2 = 2,\ x_3 = -2}.$$

Using the computer algebra systems

Solve the system using Wolfram Alpha (`http://www.wolframalpha.com/`):

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Solution:

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solve x1+x2+2*x3=1,2x1+x2+3x3=2,x1+x2+x3=3
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