Limit and continuity

Mathematics – RRMATA

Simona Fišnarová

Mendel university in Brno

Expanded set of real numbers

Definition (expanded set of real numbers)

Under an expanded set of real numbers $\mathbb{R}^*$ we understand the set $\mathbb{R}$ of all real numbers enriched by the numbers $\pm\infty$ in the following way: We set $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ and for $a \in \mathbb{R}$ we set:

$$a + \infty = \infty, \qquad a - \infty = -\infty, \qquad \infty + \infty = \infty, \qquad -\infty - \infty = -\infty$$

$$\infty \cdot \infty = -\infty \cdot (-\infty) = \infty, \qquad \infty \cdot (-\infty) = -\infty, \qquad \frac{a}{\infty} = \frac{a}{-\infty} = 0$$

$$-\infty < a < \infty, \qquad |\pm\infty| = \infty.$$

Further, for $a > 0$ we set

$$a \cdot \infty = \infty \qquad a \cdot (-\infty) = -\infty,$$

and for $a < 0$ we set

$$a \cdot \infty = -\infty \qquad a \cdot (-\infty) = \infty.$$

Another operations we define with the commutativity of the operation "$+$" and "$\cdot$".

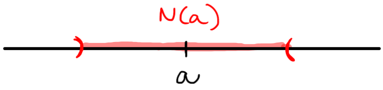
Remark (indeterminate forms)

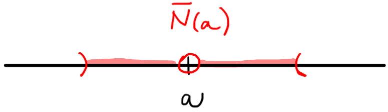
The operations "$\infty - \infty$", "$\pm\infty \cdot 0$" and "$\frac{\pm\infty}{\pm\infty}$" remain undefined. Of course, the division by a zero remains undefined as well.

Neighborhood of a point

Definition (neighborhood)

- Under the neighborhood of the point $a \in \mathbb{R}$ we understand any open interval which cotains the point a, we write $N(a)$.
- Under the reduced (also ring) neighborhood of the point a we understand the set $N(a) \setminus \{a\}$, we write $\overline{N}(a)$.
- Under the neighborhood of the point ∞ we understand the interval of the type (A, ∞) and under the neighborhood of the point $-\infty$ the interval $(-\infty, A)$. Under the reduced neighborhood of the points $\pm\infty$ we understand the same as under the neighborhood of these points.





Limit

Definition (limit of the function)

Let $a, L \in \mathbb{R}^*$ and $f : \mathbb{R} \to \mathbb{R}$. Let the function f be defined in some reduced neighborhood of the point a. We say that the function $y = f(x)$ approaches to the limit L as x approaches to a if for any (arbitrary small) neighborhood $N(L)$ of the number L there exists reduced neighborhood $\overline{N}(a)$ of the point a such that for every $x \in \overline{N}(a)$ the relation $f(x) \in N(L)$ holds.
We write

$$\lim_{x \to a} f(x) = L \tag{1}$$

or $f(x) \to L$ for $x \to a$.

Arrangement. Shortly we read (1) as "the limit of f at a is L".

One-sided neighborhood and limit

Motivation. From a geometric point of view it turns out to be interesting to distinguish the cases in which x approaches to a from the left and from the right. This gives a motivation for the following definitions.

Definition (one-sided neighborhood)

Under the right (left) neighborhood of the point $a \in \mathbb{R}$ we understand the interval of the type $[a, b)$,(or $(b, a]$, for left neighborhood), we write $N^+(a)$ $(N^-(a))$. Under the reduced right (left) neighborhood of the point a we understand the corresponding neighborhood wihtout the point a, we write $\overline{N^+}(a)$, $(\overline{N^-}(a))$

Definition (one-sided limit)

If we replace in the definition of the limit the reduced neighborhood of the point a by the reduced right neighborhood of the point a, we obtain a definition of the limit from the right. We write $\lim_{x\to a^+} f(x) = L$.
Similarly, we define also the limit from the left. In this case we write $\lim_{x\to a^-} f(x) = L$.

Remark (shortened notation)

Another (very short) notation for one-sided limits is $f(a+) = L$ for the limit from the right and $f(a-) = L$ for the limit from the left. For the two-sided limit we can write $f(a\pm)$. In several textbooks also the notation $f(a+0)$, $f(a-0)$ and $f(a\pm 0)$, respectively, is used.

Theorem (uniqueness of the limit)

The function f possesses at the point a at most one limit (or one-sided limit).

Theorem (the relationship between the limit and the one-sided limits)

The limit of the function f at the point $a \in \mathbb{R}$ exists if and only if both one-sided limits at the point a exist and are equal. More precisely: If the limits $f(a-)$ and $f(a+)$ exist and $f(a-) = f(a+)$, then the limit $f(a\pm)$ exists as well and $f(a\pm) = f(a-) = f(a+)$. If one of the one-sided limits does not exist or if $f(a-) \neq f(a+)$, then the limit $f(a\pm)$ does not exist.

Continuity

Definition (continuity at a point)

Let f be a function defined at the point $a \in \mathbb{R}$. The function f is said to be continuous at the point a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is said to be continuous from the right at the point a if $\lim_{x \to a^+} f(x) = f(a)$.
The function f is said to be continuous from the left at the point a if $\lim_{x \to a^-} f(x) = f(a)$.

Remark

According to the definition, the function is continuous at the point a if

- $f(a)$ exists,
- $\lim_{x\to a} f(x)$ exists as finite number,
- $f(a) = \lim_{x\to a} f(x)$ holds.

If the function is defined in some (at least one-sided) reduced neighborhood of the point a but at least one of the three conditions above is broken, the point $x = a$ is said to be a point of discontinuity of the function f.

Definition (continuity on an interval)

The function is said to be continuous on the open interval (a, b) if it is continuous at every point of this interval.
The function is said to be continuous on the closed interval $[a, b]$ if it is continuous on (a, b), continuous from the right at the point a and continuous from the left at the point b.

Notation. The class of all functions continuous on the interval I will be denoted by $C(I)$. If $I = (a, b)$ or $I = [a, b]$, then we write shortly $C((a, b))$ or $C([a, b])$, respectively.

Theorem (continuity of elementary functions)

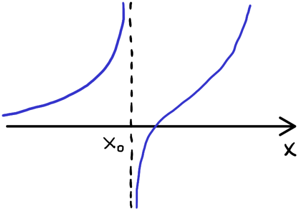
Every elementary function is continuous on its domain.

Remark (practical)

Consider an elementary function $f(x)$ and the limit of this function for $x \to a$. According to the preceding theorem, we try to substitute a for x into $f(x)$ and calculate $f(a)$ first. If this is possible, i.e. if the number a is in the domain of f, then we obtain the limit. If this fails, i.e. if $f(a)$ is not defined, we must look for another method. Hence, when speaking about elementary functions, the concept of limit gives *nothing new concerning the points from the domain*. This concept remains interesting only for the points, which are not in the natural domain of the function, but the function is defined in some ring neighborhood of these points.

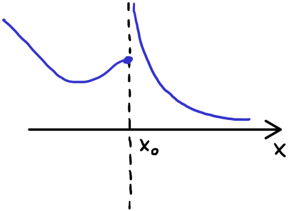
Definition (vertical asymptote)

Let f be a function and x_0 a real number. The vertical line $x = x_0$ is said to be a vertical asymptote to the graph of the function f if at lest one of the one-sided limits of the function f at the point x_0 exists and it is not a finite number.



$$\lim_{x \to x_0^+} f(x) = -\infty$$

$$\lim_{x \to x_0^-} f(x) = \infty$$



$$\lim_{x \to x_0^+} f(x) = \infty$$

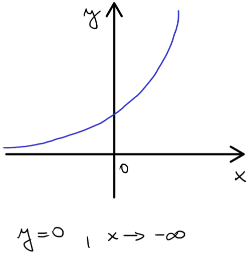
Horizontal asymptote

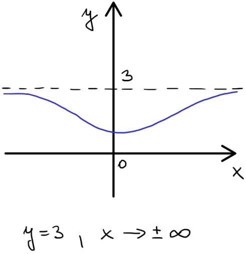
Definition (horizontal asymptote)

The line $y = L$ is said to be a horizontal asymptote to the graph of the function $f(x)$ at $+\infty$ if the limit of the function f at $+\infty$ exists and

$$\lim_{x\to\infty} f(x) = L$$

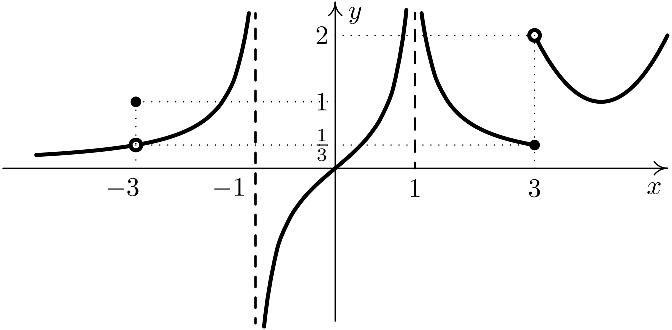
holds. In a similar way we define the *horizontal asymptote at* $-\infty$.





Example

Let the function f be given by the graph:



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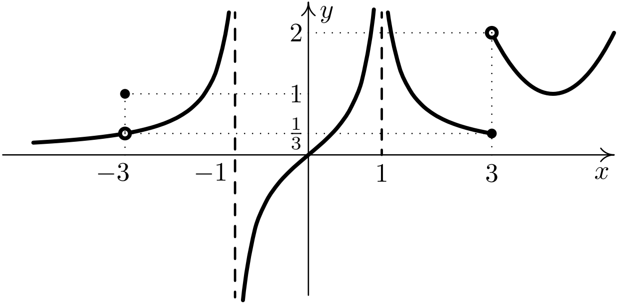

- $\lim_{x \to -\infty} f(x) = 0$

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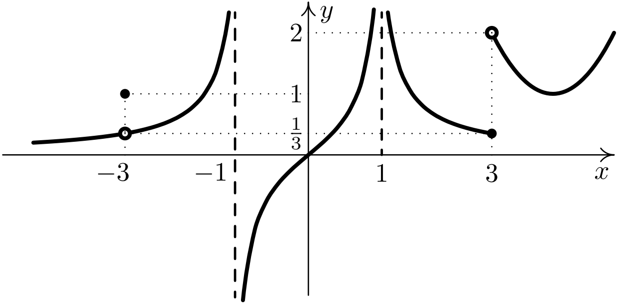

- $\lim_{x \to -\infty} f(x) = 0$
- $\lim_{x \to -3} f(x) = \frac{1}{3}$

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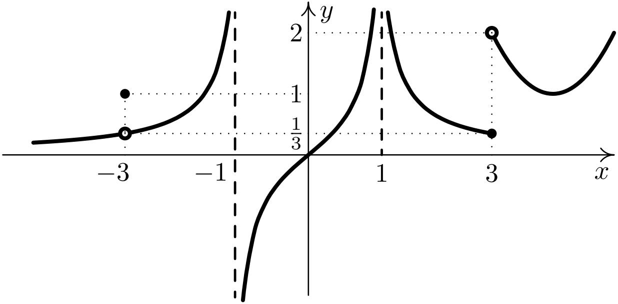

- $\lim\limits_{x\to-\infty} f(x) = 0$

- $\lim\limits_{x\to-3} f(x) = \frac{1}{3}$

- $\lim\limits_{x\to-1} f(x)$ does not exist, since

  $\lim\limits_{x\to-1^-} f(x) = \infty$, $\lim\limits_{x\to-1^+} f(x) = -\infty$

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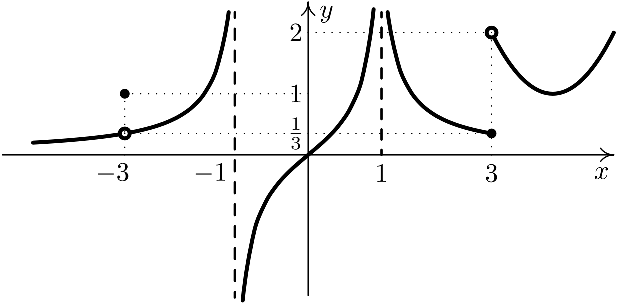

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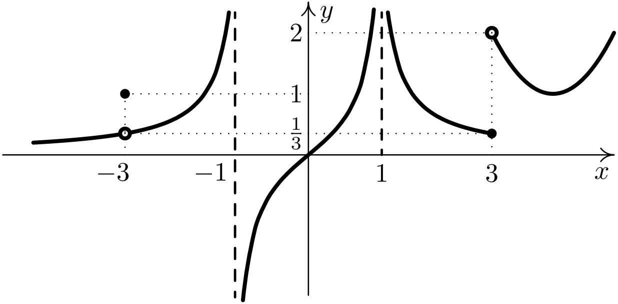

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- $\lim\limits_{x\to-1} f(x)$ does not exist, since $\lim\limits_{x\to-1^-} f(x) = \infty$, $\lim\limits_{x\to-1^+} f(x) = -\infty$
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- $\lim\limits_{x\to3} f(x)$ does not exist, since $\lim\limits_{x\to3^-} f(x) = \frac{1}{3}$, $\lim\limits_{x\to3^+} f(x) = 2$

Example

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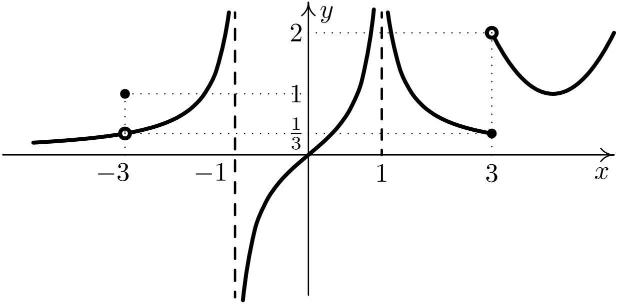

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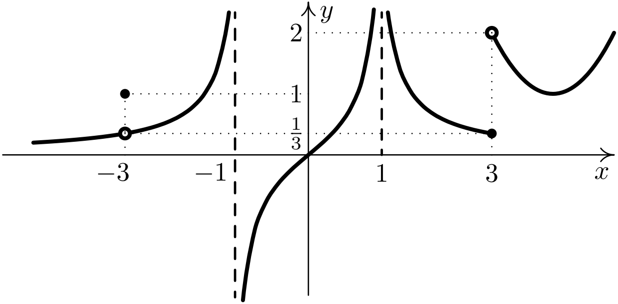

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- $\lim\limits_{x\to\infty} f(x) = \infty$

asymptotes: $x = -1$, $x = 1$ and $y = 0$ for $x \to -\infty$

Theorem (algebra of limits)

Let $a \in \mathbb{R}^*$, $f, g : \mathbb{R} \to \mathbb{R}$. The following relations hold whenever the limits on the right exist and the formula on the right is well-defined.

(2)
$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

(3)
$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

(4)
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

The same holds for one sided limits as well.

Theorem (limit of the composite function with continuous component)

Let $\lim_{x\to a} f(x) = b$ *and* $g(x)$ *be a function continuous at* b. *Then* $\lim_{x\to a} g(f(x)) = g(b)$, *i.e.*

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)).$$

The same holds for one-sided limits as well.

Theorem

Let $a \in \mathbb{R}^*$, $\lim_{x\to a} g(x) = 0$ *and* $\lim_{x\to a} f(x) = L \in \mathbb{R}^* \setminus \{0\}$. *Suppose that there exists a ring neighborhood of the point* a *such that the function* $g(x)$ *does not change its sign in this neighborhood. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} +\infty & \text{if } g(x) \text{ and } L \text{ have common sign,} \\ -\infty & \text{if } g(x) \text{ and } L \text{ have an opposite sign,} \end{cases}$$

in the neighborhood under consideration. The same holds for one sided limits as well.

Theorem (limit of the polynomial or of the rational functions at $\pm\infty$)

It holds

$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n) = \lim_{x\to\pm\infty} a_0x^n,$$

$$\lim_{x\to\pm\infty}\frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} = \lim_{x\to\pm\infty}\frac{a_0}{b_0}x^{n-m}.$$

Remark

The preceding theorem is applicable also in the cases when the rules for algebra of limits give undefined expression $\infty - \infty$ for polynomials or $\frac{\infty}{\infty}$ for rational functions!

Example (Limit of the type $\left\|\frac{a}{\pm\infty}\right\|$, $a \in \mathbb{R}$)

1. $$\lim_{x\to\infty} \frac{1}{x^2} = \frac{1}{\infty} = 0$$

2. $$\lim_{x\to 0^+} \frac{\sin x}{\ln x} = \frac{0}{-\infty} = 0$$

Example (Limit of the type $\left\| \frac{a}{0} \right\|$, $a \in \mathbb{R}^* \setminus \{0\}$)

1. $\lim\limits_{x \to 3} \dfrac{x}{(x-3)^3} = \left\| \dfrac{3}{0} \right\|$

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$$\lim_{x\to 3^+} \frac{x}{(x-3)^3} = \left\|\frac{+}{+}\right\| = \infty, \quad \lim_{x\to 3^-} \frac{x}{(x-3)^3} = \left\|\frac{+}{-}\right\| = -\infty,$$

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   $\Rightarrow \lim\limits_{x\to 3} \dfrac{x}{(x-3)^3}$ does not exist.

2. $\lim\limits_{x\to 5} \dfrac{x-7}{(x-5)^2} = \left\|\dfrac{-2}{0}\right\|$

Example (Limit of the type $\left\| \frac{a}{0} \right\|$, $a \in \mathbb{R}^* \setminus \{0\}$)

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2. $\lim\limits_{x\to 5} \dfrac{x-7}{(x-5)^2} = \left\|\dfrac{-2}{0}\right\|$ $\Rightarrow$ possible results: ∞, $-\infty$ or does not exist.
   The function $\frac{x-7}{(x-5)^5}$ is negative in a small neighborhood of $x = 5$, hence

$$\lim_{x\to 5} \frac{x-7}{(x-5)^2} = \left\|\frac{-}{+}\right\| = -\infty.$$

Example (Limit of the polynomial and the rational function)

1. $\lim\limits_{x \to -\infty} (2x^3 - 4x^2 + x - 2)$

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2. $\lim\limits_{x \to \infty} \dfrac{x^4 + 3x^2 - 2x}{3x^4 + 5}$

Example (Limit of the polynomial and the rational function)

1. $\lim\limits_{x\to-\infty}(2x^3-4x^2+x-2)=\lim\limits_{x\to-\infty}2x^3=-\infty.$

2. $\lim\limits_{x\to\infty}\dfrac{x^4+3x^2-2x}{3x^4+5}=\lim\limits_{x\to\infty}\dfrac{x^4}{3x^4}$

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3. $\lim\limits_{x\to-\infty}\dfrac{2x^6+3x^4-2}{3x^2+x-1}$

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3. $\lim_{x \to -\infty} \frac{2x^6 + 3x^4 - 2}{3x^2 + x - 1} = \lim_{x \to -\infty} \frac{2x^6}{3x^2} = \lim_{x \to -\infty} \frac{2}{3}x^4$

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2. $$\lim_{x\to\infty}\frac{x^4+3x^2-2x}{3x^4+5}=\lim_{x\to\infty}\frac{x^4}{3x^4}=\frac{1}{3}.$$

3. $$\lim_{x\to-\infty}\frac{2x^6+3x^4-2}{3x^2+x-1}=\lim_{x\to-\infty}\frac{2x^6}{3x^2}=\lim_{x\to-\infty}\frac{2}{3}x^4=\infty.$$

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4. $\lim_{x\to\infty} \frac{5x^2 - 3x - 2}{3x^3 + x^2 + 2} = \lim_{x\to\infty} \frac{5x^2}{3x^3}$

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Theorems on continuous functions

Theorem (Weierstrass)

Let f be a function defined and continuous on $[a, b]$. Then the function f is bounded and takes on an absolute maximum and an absolute minimum on the interval $[a, b]$, i.e. there exist numbers $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

Theorem (Bolzano, the 1st Bolzano's theorem)

Let f be a function defined and continuous on $[a, b]$. If $f(a).f(b) < 0$ holds (i.e. the values $f(a)$ and $f(b)$ have different signs), then there exists a zero of the function f on the interval (a, b), i.e. there exists $c \in (a, b)$ such that $f(c) = 0$.

Theorem (Bolzano, the 2nd Bolzano's theorem)

Let f be a function defined and continuous on $[a, b]$. Let m and M be absolute minimum and absolute maximum of the funciton f on the interval $[a, b]$, respectively. Then for every y_0 between m and M there exists at least one x_0 with property $f(x_0) = y_0$.

Bisection method

The bisection method is a simple method for approximation of the zeros of continuous functions. Given a continuous function $f(x)$ and real numbers $a, b \in \mathbb{R}$, suppose that $f(a)f(b) < 0$ holds. According to the first Bolzano theorem, there exists $\tilde{c} \in [a, b]$ such that $f(\tilde{c}) = 0$. Consider the point $c = \frac{b+a}{2}$ and the value $f(c)$. One of the relations

$$f(a)f(c) < 0 \quad \text{or} \quad f(c)f(b) < 0 \quad \text{or} \quad f(c) = 0$$

holds. Omitting the last possibility (in this case $x = c$ is an exact zero of the function $f(x)$), we see that one of the intervals (a, c) and (b, c) contains a zero of the function $f(x)$. When seeking a zero, we can focus our attention to the appropriate left half or right half of the interval (a, b). Hence the localisation of the zero is somewhat better: the length of the interval with change of sign is one half to the original length and hence the accuracy is two times better. Following this idea we can, after a finite number of steps, obtain the value of the zero with an arbitrary precision.

Using the computer algebra systems

Wolfram Alpha:

http://www.wolframalpha.com/

Example

Find the limits:

$$\lim_{x\to\infty} \frac{x^5 - 3x^4 + x - 2}{2x^3 + 4}, \quad \lim_{x\to 0^+} \frac{\ln x}{x}, \quad \lim_{x\to 1} \frac{x+2}{x-1}.$$

Solution:

```
lim (x^5-3x^4+x-2)/(2x^3+4) as x->infty

lim (ln x)/x as x->0+

lim ((x+2)/(x-1)) as x->1
```