

Indefinite integral

Mathematics – FRDIS

MENDELU

Antiderivative and indefinite integral

Definition (Antiderivative)

Let f and F be functions defined on an open interval I . If

$$F'(x) = f(x) \quad \text{for every } x \in I,$$

then the function F is called an **antiderivative** of f on the interval I .

It follows from the definition: If F is an antiderivative of f , then $F + c$ (where c is any real constant), is also an antiderivative of f . Indeed,

$$[F(x) + c]' = F'(x) + c' = f(x) + 0 = f(x).$$

This means that if there exists an antiderivative of f on the given interval, then there exist infinitely many antiderivatives of f on this interval.

Example

For example, the following functions are antiderivatives of $y = x^3$:

$$y = \frac{x^4}{4}, \quad y = \frac{x^4}{4} + 3, \quad y = \frac{x^4}{4} - 7.$$

Indeed,

$$\left(\frac{x^4}{4}\right)' = x^3, \quad \left(\frac{x^4}{4} + 3\right)' = x^3, \quad \left(\frac{x^4}{4} - 7\right)' = x^3.$$

Evidently any function of the form $y = \frac{x^4}{4} + c$, where $c \in \mathbb{R}$, is an antiderivative of $y = x^3$.

Question: Are the functions of the form $y = \frac{x^4}{4} + c$ the only antiderivatives of $y = x^3$ or can we find another antiderivative?

The following theorem gives answer to this question.

Theorem (Uniqueness of antiderivative)

Let F and G be two antiderivatives of f on an interval I . Then there exists a constant $c \in \mathbb{R}$ such that $G(x) = F(x) + c$ for every $x \in I$.

The previous theorem says that any two antiderivatives of the same function f differ only by an additive constant.

Definition (Indefinite integral)

The set of all antiderivatives of f is denoted $\int f(x) dx$ and it is called the **indefinite integral**. We write

$$\int f(x) dx = F(x) + c,$$

where F is any antiderivative of f and c is any real constant.

The symbol \int is called the **integral sign**, the function f is called **integrand**, the constant c is called the **constant of integration**. The dx is part of integral notation and indicates the variable involved. To **integrate** f means to find all antiderivatives of f . The function which has an antiderivative (indefinite integral) is called **integrable**.

It holds:

$$\left(\int f(x) dx \right)' = f(x) \quad \text{and} \quad \int (F(x))' dx = F(x) + c$$

Theorem (Sufficient condition for integrability)

Let f be a continuous function on I . Then there exists an antiderivative of f on I .

- There exist functions which are not continuous and have antiderivatives.
- There exist functions which are continuous (hence they have antiderivatives), but the antiderivatives cannot be expressed using the elementary functions. The antiderivatives of these functions, such as

$$\int e^{-x^2} dx, \quad \int \frac{e^x}{x} dx, \\ \int \frac{\sin x}{x} dx, \quad \int \frac{\cos x}{x} dx, \quad \int \sin x^2 dx, \quad \int \cos x^2 dx$$

are called **higher transcendental functions**.

Basic Formulas for integration

$$\int 0 dx = c$$

$$\int 1 dx = x + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \frac{1}{x} dx = \ln |x| + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + c$$

$$\int \frac{1}{\sin^2 x} dx = -\operatorname{cotg} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$$

$$\int \frac{1}{x^2+1} dx = \operatorname{arctg} x + c$$

- We usually write $\int dx$ instead of $\int 1 dx$.
- Sometimes we write $\int \frac{dx}{f(x)}$ instead of $\int \frac{1}{f(x)} dx$.

Basic rules for integration

Theorem (Basic rules for integration)

Let f and g be functions, which are integrable on an interval I and let $c \in \mathbb{R}$. Then the following formulas hold on I :

$$\begin{aligned}\int [f(x) \pm g(x)] dx &= \int f(x) dx \pm \int g(x) dx \\ \int c \cdot f(x) dx &= c \int f(x) dx\end{aligned}$$

We don't have any general rule for integration products and quotients of functions and for the composite function !!!

Example (Integration of power functions and polynomials)

$$\textcircled{1} \int x^3 dx = \frac{x^4}{4} + c$$

$$\textcircled{2} \int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \sqrt{x^3} + c$$

$$\textcircled{3} \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} = -\frac{1}{3x^3} + c$$

$$\textcircled{4} \int (x^4 + 2x^3 + x - 2) dx = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^2}{2} - 2x + c$$

Example (Basic formulas and simplification of integrand)

- ① Multiplication of the power functions

$$\int x^2 \sqrt{x} dx = \int x^{\frac{5}{2}} dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{2}{7} x^3 \sqrt{x} + c$$

- ② Decomposition of the numerator of a fraction

$$\begin{aligned} \int \frac{x+3}{x^2} dx &= \int \left(\frac{x}{x^2} + \frac{3}{x^2} \right) dx = \int \left(\frac{1}{x} + 3x^{-2} \right) dx \\ &= \ln|x| + 3 \cdot \frac{x^{-1}}{-1} = \ln|x| - \frac{3}{x} + c \end{aligned}$$

- ③ Decomposition of the improper rational function into a sum of a polynomial and the proper rational function. (We divide the numerator by the denominator.)

$$\int \frac{x^4}{x^2+1} dx = \int \left(x^2 - 1 + \frac{1}{x^2+1} \right) dx = \frac{x^3}{3} - x + \arctg x + c$$

Example (Using formulas for trigonometric functions)

We use mainly the following formulas:

$$\operatorname{tg} x = \frac{\sin x}{\cos x}, \quad \operatorname{cotg} x = \frac{\cos x}{\sin x}, \quad \sin^2 x + \cos^2 x = 1$$

$$\begin{aligned} \int \operatorname{tg}^2 x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \left(\frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} \right) \, dx \\ &= \int \left(\frac{1}{\cos^2 x} - 1 \right) \, dx = \operatorname{tg} x - x + c \end{aligned}$$

Method of substitution

Theorem (The 1 st method of substitution, $t = \varphi(x)$)

Let $f(t)$ be a function which is continuous on I and $\varphi(x)$ be a function having continuous derivative on J . Next suppose that $\varphi(x) \in I$ for every $x \in J$. Then the function $f[\varphi(x)]\varphi'(x)$ is continuous on J and it holds on this interval:

$$\int f[\varphi(x)]\varphi'(x) \, dx = \int f(t) \, dt,$$

where we substitute $t = \varphi(x)$ on the right-hand side.

- The method can be used if the integrand is of the form “**composite function** $f[\varphi(x)]$ times **the derivative of the interior function** $\varphi'(x)$ ”
- We write a new variable t instead of the interior function $\varphi(x)$ and we write dt instead of $\varphi'(x) \, dx$.
- Let F be an antiderivative of f , then we proceed as follows:

$$\int f[\varphi(x)]\varphi'(x) \, dx = \left| \begin{array}{l} t = \varphi(x) \\ dt = \varphi'(x) \, dx \end{array} \right| = \int f(t) \, dt = F(t) + c = F[\varphi(x)] + c$$

Example (The 1 st method of substitution)

$$\textcircled{1} \int \sin(3x + 2) dx = \left| \begin{array}{l} t = 3x + 2 \\ dt = 3 dx \\ dx = \frac{1}{3} dt \end{array} \right| = \frac{1}{3} \int \sin t dt = -\frac{1}{3} \cos t + c \\ = -\frac{1}{3} \cos(3x + 2) + c$$

$$\textcircled{2} \int \sin^2 x \cos x dx = \int (\sin x)^2 \cdot \cos x dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int t^2 dt \\ = \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c$$

$$\textcircled{3} \int x \cdot e^{1-x^2} dx = \left| \begin{array}{l} t = 1 - x^2 \\ dt = -2x dx \\ x dx = -\frac{1}{2} dt \end{array} \right| = -\frac{1}{2} \int e^t dt = -\frac{1}{2} e^t + c = -\frac{1}{2} e^{1-x^2} + c$$

$$\textcircled{4} \int x^3 \cdot \sqrt{x^4 + 1} dx = \left| \begin{array}{l} t = x^4 + 1 \\ dt = 4x^3 dx \\ x^3 dx = \frac{1}{4} dt \end{array} \right| = \frac{1}{4} \int \sqrt{t} dt = \frac{1}{4} \int t^{\frac{1}{2}} dt \\ = \frac{1}{4} \cdot \frac{2}{3} t^{\frac{3}{2}} + c = \frac{1}{6} \sqrt{t^3} + c = \frac{1}{6} \sqrt{(x^4 + 1)^3} + c$$

Composite function with linear interior function

Let F be an antiderivative of f on an interval I . Then for $ax + b \in I$ we have:

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + c.$$

Example

$$\int \sin(5x + 1) dx$$

- Using substitution:

$$\int \sin(5x + 1) dx = \left| \begin{array}{l} t = 5x + 1 \\ dt = 5 dx \\ dx = \frac{1}{5} dt \end{array} \right| = \frac{1}{5} \int \sin t dt = -\frac{1}{5} \cos t + c = \\ -\frac{1}{5} \cos(5x + 1) + c$$

- Using the above formula: $ax + b = 5x + 1$

$$f(x) = \sin(x) \implies F(x) = -\cos x$$

$$f(ax + b) = \sin(5x + 1) \implies F(ax + b) = -\cos(5x + 1)$$

$$\implies \int \sin(5x + 1) dx = -\frac{1}{5} \cos(5x + 1) + c$$

Example

$$\textcircled{1} \int \cos 2x \, dx = \frac{1}{2} \sin 2x + c$$

$$\textcircled{2} \int (3 - 5x)^6 \, dx = -\frac{1}{5} \cdot \frac{1}{7} (3 - 5x)^7 = -\frac{1}{35} (3 - 5x)^7 + c$$

$$\textcircled{3} \int \frac{1}{2x - 3} \, dx = \frac{1}{2} \ln |2x - 3| + c$$

$$\textcircled{4} \int e^{2-x} \, dx = -e^{2-x} + c$$

$$\textcircled{5} \int \sin \frac{x}{2} \, dx = \int \sin \left(\frac{1}{2}x \right) \, dx = -2 \cos \frac{x}{2} + c$$

The integrals can be solved using the substitutions:

$$t = 2x, \quad t = 3 - 5x, \quad t = 2x - 3, \quad t = 2 - x, \quad t = \frac{x}{2}.$$

f'/f

$$\int \frac{f'(x)}{f(x)} \, dx = \left| \begin{array}{l} t = f(x) \\ dt = f'(x) \, dx \end{array} \right| = \int \frac{1}{t} \, dt = \ln |t| + c = \ln |f(x)| + c$$

$$\Rightarrow \boxed{\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c}$$

Example

$$\textcircled{1} \int \frac{2x}{x^2 - 3} \, dx = \ln |x^2 - 3| + c$$

$$\textcircled{2} \int \frac{3x}{x^2 - 3} \, dx = \frac{3}{2} \int \frac{2x}{x^2 - 3} \, dx = \frac{3}{2} \ln |x^2 - 3| + c$$

$$\textcircled{3} \int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{-\sin x}{\cos x} \, dx = -\ln |\cos x| + c$$

$$\textcircled{4} \int \operatorname{cotg} x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| + c$$

Trigonometric substitution

The integral of the form $\int R(\sin x, \cos x) dx$, where R is a rational function, can be transformed (with using special substitutions) into the integral from a rational function. Special cases:

$$\textcircled{1} \int R(\sin x) \cos x dx \Rightarrow t = \sin x$$

$$\textcircled{2} \int R(\cos x) \sin x dx \Rightarrow t = \cos x$$

Remark

- Functions of the type $R(\sin x, \cos x)$: $\frac{\sin x \cos^2 x}{\sin^2 x + \cos x}$, $\frac{\cos^3 x + 1}{\sin^2 x}$
- Functions of the type $R(\sin x)$: $\frac{\sin x + 2}{\sin^2 x}$, $\frac{2 \sin x}{\sin x - 1}$
- Functions of the type $R(\cos x)$: $\frac{\cos x}{\cos^2 x + 3}$, $\frac{\cos^2 x + \cos x}{\cos x + 1}$

Example

$$\begin{aligned} \textcircled{1} \int \frac{\sin x \cos x}{\sin^2 x + 2} dx &= \int \frac{\sin x}{\sin^2 x + 2} \cos x dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{t}{t^2 + 2} dt \\ &= \frac{1}{2} \int \frac{2t}{t^2 + 2} dt = \frac{1}{2} \ln(t^2 + 2) + c = \frac{1}{2} \ln(\sin^2 x + 2) + c \end{aligned}$$

$$\begin{aligned} \textcircled{2} \int \frac{\sin x}{\cos^3 x} dx &= \int \frac{1}{\cos^3 x} \sin x dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x dx \\ \sin x dx = -dt \end{array} \right| = - \int \frac{1}{t^3} dt \\ &= - \int t^{-3} dt = -\frac{t^{-2}}{-2} + c = \frac{1}{2t^2} + c = \frac{1}{2 \cos^2 x} + c \end{aligned}$$

$$\begin{aligned} \textcircled{3} (*) \int \frac{\sin^3 x}{\cos x + 3} dx &= \int \frac{\sin^2 x}{\cos x + 3} \sin x dx = \int \frac{1 - \cos^2 x}{\cos x + 3} \sin x dx \\ &= \left| \begin{array}{l} t = \cos x \\ dt = -\sin x dx \\ \sin x dx = -dt \end{array} \right| = \int \frac{t^2 - 1}{t + 3} dt = \int \left(t - 3 + \frac{8}{t + 3} \right) dt \\ &= \frac{t^2}{2} - 3t + 8 \ln |t + 3| + c = \frac{\cos^2 x}{2} - 3 \cos x + 8 \ln |\cos x + 3| + c \end{aligned}$$

Theorem (The 2 nd method of substitution, $x = \varphi(t)$)

Let $f(x)$ be a function which is continuous on I and $\varphi(t)$ be a function having continuous and nonzero derivative on J . Next suppose that $\varphi(J) = I$. Then it holds on I :

$$\int f(x) dx = \int f[\varphi(t)]\varphi'(t) dt,$$

where we substitute $t = \varphi^{-1}(x)$ on the right-hand side and where φ^{-1} is the inverse function to φ .

- We write $\varphi(t)$ instead of x and $\varphi'(t) dt$ instead of dx . The composite function obtained in the integral on the right-hand side seems to be more complicated than the original one, but in some particular cases it can be easier to integrate this composite function.
- We proceed as follows:

$$\int f(x) dx = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) dt \end{array} \right| = \int f[\varphi(t)]\varphi'(t) dt = F(t) + c = F[\varphi^{-1}(x)] + c,$$

where $F(t)$ is an antiderivative of $f[\varphi(t)]\varphi'(t)$.

Integration of irrational function

We can eliminate roots in the integral

$$\int R(x, \sqrt[n_1]{ax+b}, \sqrt[n_2]{ax+b}, \dots) dx,$$

where R is a rational function. We use the substitution

$$t = \sqrt[s]{ax+b}, \quad \text{where } s \text{ is the least common multiple of } n_1, n_2, \dots$$

Then:

$$t = \sqrt[s]{ax+b} \implies ax+b = t^s$$

$$x = \frac{1}{a}t^s - \frac{b}{a}$$

$$dx = \frac{1}{a}st^{s-1} dt$$

Example

$$\begin{aligned} \textcircled{1} \int \frac{x}{\sqrt{x+1}} dx &= \left| \begin{array}{l} t = \sqrt{x+1} \Rightarrow x = t^2 - 1 \\ dx = 2t dt \end{array} \right| = \int \frac{t^2 - 1}{t} 2t dt \\ &= 2 \int (t^2 - 1) dt = 2 \left(\frac{t^3}{3} - t \right) + c = \frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1} + c \end{aligned}$$

$$\begin{aligned} \textcircled{2} \int \frac{2}{\sqrt{x}(x+1)} dx &= \left| \begin{array}{l} t = \sqrt{x} \Rightarrow x = t^2 \\ dx = 2t dt \end{array} \right| = \int \frac{2}{t(t^2+1)} 2t dt \\ &= 4 \int \frac{1}{t^2+1} dt = 4 \operatorname{arctg} t + c = 4 \operatorname{arctg} \sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \textcircled{3} \int \frac{\sqrt{x}}{\sqrt[3]{x+1}} dx &= \left| \begin{array}{l} t = \sqrt[6]{x} \Rightarrow x = t^6 \\ dx = 6t^5 dt \end{array} \right| = \int \frac{t^3}{t^2+1} 6t^5 dt = 6 \int \frac{t^8}{t^2+1} dt \\ &= 6 \int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{t^2+1} \right) dt = 6 \left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \operatorname{arctg} t \right) + c \\ &= \frac{6\sqrt[6]{x^7}}{7} - \frac{6\sqrt[6]{x^5}}{5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6 \cdot \operatorname{arctg} \sqrt[6]{x} + c \end{aligned}$$

Integration by parts

Theorem (Integration by parts formula)

Let u and v be function having continuous derivatives on an interval I . Then the following formula holds on I :

$$\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx.$$

- ① Integration by parts formula follows from the rule for differentiation of the product of two functions.
- ② When using the integration by parts formula we need:
 - to differentiate the function u ,
 - to integrate the function v' . This can be a problem!!!

Moreover, for an effective application of this formula we need the product $u'v$ (which appears in the integral on the right-hand side) to be "simpler in sense of integration" than the original uv' .

Typical integrals for using integration by parts

Let P be a polynomial.

①

$$\int P(x)e^{ax+b} dx, \quad \int P(x) \sin(ax + b) dx, \quad \int P(x) \cos(ax + b) dx$$

In these cases we differentiate the polynomial and we integrate the exponential (or trigonometric) function.

②

$$\int P(x) \ln^m(ax + b) dx$$
$$\int P(x) \operatorname{arctg}(ax + b) dx, \quad \int P(x) \operatorname{arccotg}(ax + b) dx$$
$$\int P(x) \arcsin(ax + b) dx, \quad \int P(x) \arccos(ax + b) dx$$

In these cases we integrate the polynomial and we differentiate the logarithmic or the cyclometric function. The case $P(x) = 1$ is also included.

Example (Integration by parts I)

$$\textcircled{1} \int x \cos x dx = \left| \begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array} \right| = x \sin x - \int \sin x dx$$
$$= x \sin x + \cos x + c$$

$$\textcircled{2} \int (x^2 + 1)e^{-x} dx = \left| \begin{array}{ll} u = x^2 + 1 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{array} \right|$$
$$= (x^2 + 1)(-e^{-x}) - \int 2x(-e^{-x}) dx$$
$$= -e^{-x}(x^2 + 1) + 2 \int xe^{-x} dx = \left| \begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{array} \right|$$
$$= -e^{-x}(x^2 + 1) + 2 \left[x(-e^{-x}) - \int 1 \cdot (-e^{-x}) dx \right]$$
$$= -e^{-x}(x^2 + 1) - 2xe^{-x} + 2 \int e^{-x} dx$$
$$= -e^{-x}(x^2 + 1) - 2xe^{-x} - 2e^{-x} + c = -e^{-x}(x^2 - 2x + 3) + c$$

Example (Integration by parts II)

$$\textcircled{1} \int \ln x \, dx = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right| = x \ln x - \int \frac{1}{x} \cdot x \, dx$$
$$= x \ln x - \int dx = x \ln x - x + c$$

$$\textcircled{2} \int x \operatorname{arctg} x \, dx = \left| \begin{array}{ll} u = \operatorname{arctg} x & v' = x \\ u' = \frac{1}{x^2+1} & v = \frac{x^2}{2} \end{array} \right| = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{x^2+1} \, dx$$
$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2+1} \right) dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} (x - \operatorname{arctg} x) + c$$

Using the computer algebra systems

Wolfram Alpha:

<http://www.wolframalpha.com/>

Mathematical Assistant on Web (MAW):

wood.mendelu.cz/math/maw-html/index.php?lang=en&form=integral

Example

Using the Wolfram Alpha find the integral

$$\int \ln x \, dx.$$

Solution:

integrate ln x dx