Functions – basic concepts

Mathematics – RRMATA

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Functions

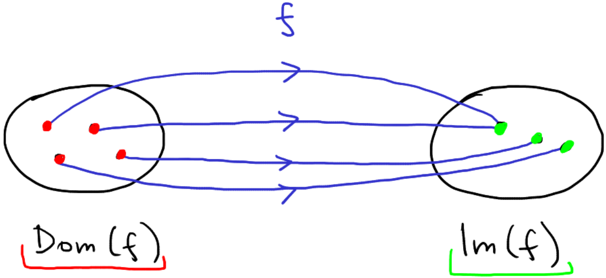
Definition (function)

Let A and B be nonempty sets of real numbers.
Let f be a rule which associates each element x of the set A with exactly one element y of the set B. The rule f is said to be a function defined on A. We write $f : A \to B$. If f associates x with y, we write $y = f(x)$.

- The variable x is customary called an independent variable and y is called a dependent variable.
- The set A is called a domain of the function f and denoted by $\mathrm{Dom}(f)$.
- The set B is a target set. The subset of all that elements y of the set B which are generated by the elements from $\mathrm{Dom}(f)$ is called an image (or range) of the function f and denoted by $\mathrm{Im}(f)$.

Examples of functions:

$$y = \sin x,\ y = \ln x,\ y = 3^x,\ y = x^4 + 2x,\ y = \frac{\sqrt{x+1}}{x}.$$


f
Dom(f)
Im(f)

Definition (graph)

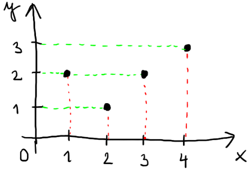
Let f be a function.

A graph of the function f is the set of all of the points in the plane $[x, y] \in \mathbb{R}^2$ with property $y = f(x)$.

If $0 \in \mathrm{Dom}(f)$, then the value $f(0)$ is well-defined and the point point $[0, f(0)]$ on the graph is called an y-intercept of the graph.

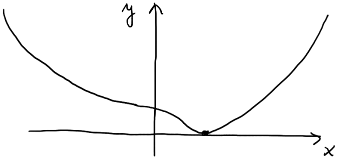
The number $x \in \mathrm{Dom}(f)$ which satisfies $f(x) = 0$ is called a root or a zero of the function f.

If the number x_0 is a zero of the function f, then the point $[x_0, 0]$ on the graph is called an x-intercept of the graph.



$\mathrm{Dom}(f) = \{1, 2, 3, 4\}$

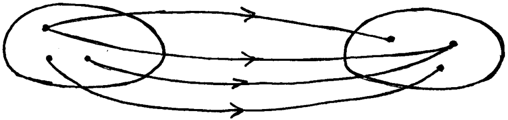
$\mathrm{Im}(f) = \{1, 2, 3\}$



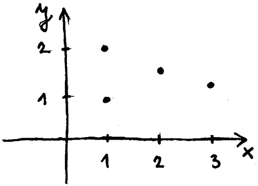
$\mathrm{Dom}(f) = \mathbb{R}$

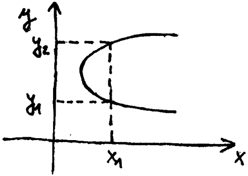
$\mathrm{Im}(f) = \langle 0, \infty)$

- This mapping is not a function:



- These points and a curve are not graphs of functions:





Basic elementary functions

Several functions which are common in applications have special names. This concerns especially the following functions:

1. Power function $y = x^\alpha$, $\alpha \in \mathbb{R}$
2. Exponential function $y = a^x$, $a > 0,\ a \neq 1$
3. Logarithmic function $y = \log_a x$, $a > 0,\ a \neq 1$
4. Trigonometric functions $y = \sin x$, $y = \cos x$, $y = \mathrm{tg} x$, $y = \mathrm{cotg} x$
5. Inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$, $y = \mathrm{arctg} x$, $y = \mathrm{arccotg} x$

The functions in the preceding list are called basic elementary functions.

- A sum of constant multiples of power functions with a nonnegative integer exponent is called a polynomial, e.g. $y = 2x^3 - 4x + 1$ is a polynomial. The highest power in the polynomial is called a degree of the polynomial.
- The quotient of two polynomials is called a rational function, e.g. $y = \frac{x}{x^3 - 2x + 1}$ is a rational function.

Elementary functions

The function f which can be represented by a single formula $y = f(x)$, where the expression on the right-hand side is made up of basic elementary functions and constants by means of a finite number of the operations addition, subtraction, multiplication, division and composition is called an elementary function. In the exercises we will almost exclusively work with elementary functions, since this is sufficient to describe most of the phenomena in scientific and economical applications.

Remark

In calculus we are interested especially in the composition of the basic elementary functions. Hence $y = \ln(x+1)$, $y = \arcsin\sqrt{x}$ and $y = \sin\frac{x}{2}$ are treated as composite functions. On the other hand, the basic elementary functions like $y = e^x$ and $y = \mathrm{tg} x$ are not composite functions.

Composite function

Definition (composite function)

Let $f : A \to B$ and $g : B \to C$ be functions. Under a composite function $g \circ f$ (read "g of f") we understand the function defined for every $x \in A$ by the relation

$$(g \circ f)(x) = g(f(x)).$$

The function f is said to be an inside function and g an outside function of the composite function $g \circ f$.

Example

$f : y = x^2, \quad g : y = \sin x$

- $g \circ f : y = g(f(x)) = \sin x^2$
- $f \circ g : y = f(g(x)) = \sin^2 x$

The composite function may consist of more than two functions, e.g.:

$$(h \circ g \circ f)(x) = h(g(f(x))).$$

Example

$f : y = x^2, \quad g : y = \sin x, \quad h : y = \ln x$

- $h \circ g \circ f : y = h(g(f(x))) = \ln \sin x^2$
- $f \circ g \circ h : y = f(g(h(x))) = \sin^2 \ln x$
- $f \circ h \circ g : y = f(h(g(x))) = \ln^2 \sin x$
- $g \circ f \circ h : y = g(f(h(x))) = \sin \ln^2 x$

Basic properties of functions

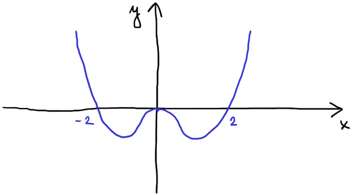
Definition (odd and even function)

Let f be a function with domain $\mathrm{Dom}(f)$. Let for every $x \in \mathrm{Dom}(f)$ also $-x \in \mathrm{Dom}(f)$.

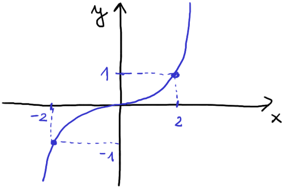
1. The function f is said to be even, if $f(-x) = f(x)$ holds for every $x \in \mathrm{Dom}(f)$.
2. The function f is said to be odd, if $f(-x) = -f(x)$ holds for every $x \in \mathrm{Dom}(f)$.

Arrangement: The function is said to have a parity if it is either odd or even.

Graph of an even function:



Graph of an odd function:



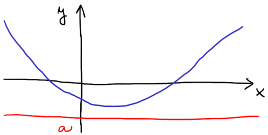
Definition (boundedness)

Let f be a function and $M \subseteq \mathrm{Dom}(f)$ be a subset of the domain of the function f.

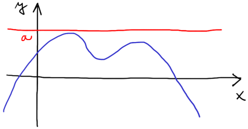
1. The function f is said to be bounded from below on the set M if there exists a real number a with the property $a \leq f(x)$ for all $x \in M$.
2. The function f is said to be bounded from above on the set M if there exists a real number a with the property $a \geq f(x)$ for all $x \in M$.
3. The function f is said to be bounded on the set M if it is bounded from both below and above on the set M.

If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

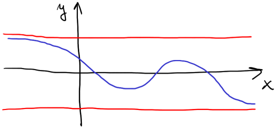
Graph of a function bounded from below:



Graph of a function bounded from above:



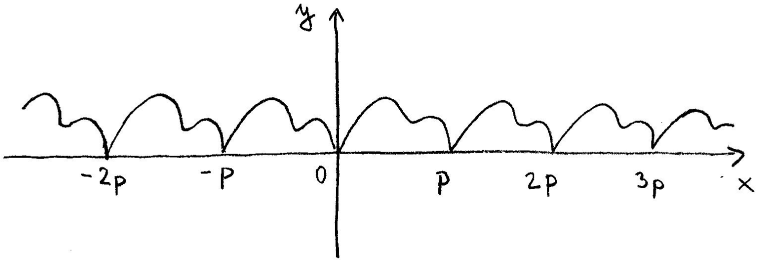
Graph of a bounded function:



Definition (periodic function)

Let $p \in \mathbb{R}$, $p > 0$. A function f is called periodic with the period p, if for every $x \in \mathrm{Dom}(f)$ holds

$$x \pm p \in \mathrm{Dom}(f) \quad \text{and} \quad f(x+p) = f(x) = f(x-p).$$



Motivation. Now we will distinguish the classes of functions which preserve or reverse inequalities. This yields the following classes of functions.

Definition (monotonicity)

Let f be a function and $M \subseteq \mathrm{Dom}(f)$ be a subset of its domain.

1. The function f is said to be increasing on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) < f(x_2)$ holds.
2. The function f is said to be decreasing on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) > f(x_2)$ holds.
3. The function f is said to be (strictly) monotone on the set M if it is either increasing or decreasing on M.

If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

Motivation. We start with an obvious implication which holds for every well-defined function f.

$$(1) \qquad x_1 = x_2 \implies f(x_1) = f(x_2)$$

The equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent, if the implication in (1) can be reversed. It is clear that this implication *cannot* be reversed if two mutually different values $x_1 \neq x_2$ in the domain of f may generate the same value of the function f. A property which excludes this possibility is introduced in the following definition.

Definition (one-to-one function)

Let f be a function and let $M \subseteq \mathrm{Dom}(f)$ be a subset of the domain of the function f.
The function f is said to be one-to-one function on the set M if there exists no pair of different elements x_1, x_2 of the set M which are associated with the same element y of the image $\mathrm{Im}(f)$.
If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

Remark (one-to-one functions)

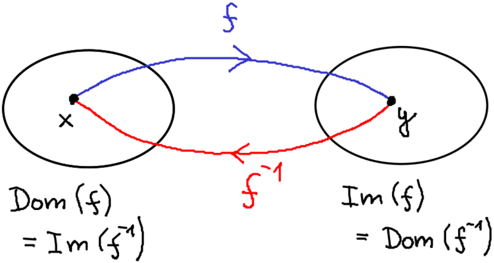
Mathematically formulated, for the one-to-one functions the following implication holds

$$f(x_1) = f(x_2) \implies x_1 = x_2. \tag{2}$$

This implication means that if the function f is one-to-one, then we can remove (or more precisely un-apply) this function from both sides of the relation $f(x_1) = f(x_2)$ and conclude the equivalent relation $x_1 = x_2$. This property can be used when solving nonlinear equations, since this is an exact description of the fact that the equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent.

Definition (inverse function)

Let $f : A \to B$ be an one-to-one function. The rule which associates every x with the number y satisfying $f(y) = x$ defines a function on B (really, for every $x \in \mathrm{Im}(f) \subseteq B$ there exists only one y satisfying $f(y) = x$). This function is said to be an inverse function to the function f. The inverse function to the function f is denoted by f^{-1}.



Remark

The inverse function to some of the basic elementary function is sometimes simply a name of another elementary function. Table 1 summarizes the basic elementary function and their inverses. The property to be an inverse function is mutual. If, for example, the $\arcsin(\cdot)$ function is inverse to the $\sin(\cdot)$ function, then also the $\sin(\cdot)$ function is an inverse to the $\arcsin(\cdot)$ function.

The function $y = f(x)$	The inverse function $y = f^{-1}(x)$
$y = \sqrt{x}$	$y = x^2$, $x > 0$
$y = x^2$, $x > 0$	$y = \sqrt{x}$
$y = e^x$	$y = \ln x$
$y = \ln x$	$y = e^x$
$y = a^x$	$y = \log_a x$
$y = \sin x$, $x \in [-\pi/2, \pi/2]$	$y = \arcsin x$
$y = \cos x$, $x \in [0, \pi]$	$y = \arccos x$
$y = \mathrm{tg} x$, $x \in [-\pi/2, \pi/2]$	$y = \mathrm{arctg} x$

Table: Inverse functions to the basic elementary functions

Remark (nonlinear equations)

The inverse function allows an alternative interpretation in terms of solutions of nonlinear equations: If there exists an inverse f^{-1} of the function f and if the inverse function is defined in some x, then the nonlinear equation with the unknown y

$$f(y) = x \tag{3}$$

has exactly one solution. This solution is given by the formula

$$y = f^{-1}(x). \tag{4}$$

An alternative method how to obtain (4) from (3) is to write the right-hand side of (3) in the form $f(f^{-1}(x))$ and conclude (4) via implication (2).

Remark (summing up remark)

Finally, we summarize all the properties of the functions introduced in this chapter. We use the notation which is usefull when solving equations and (or) inequalities.

$$a = b \overset{f \text{ is one-to-one}}{\iff} f(a) = f(b)$$

$$a < b \overset{f \text{ is increasing}}{\iff} f(a) < f(b) \qquad a \leq b \overset{f \text{ is increasing}}{\iff} f(a) \leq f(b)$$

$$a < b \overset{f \text{ is decreasing}}{\iff} f(a) > f(b) \qquad a \leq b \overset{f \text{ is decreasing}}{\iff} f(a) \geq f(b)$$

If the function f is one-to-one, then for every $y \in \mathrm{Im}(f)$ there exists a unique solution x of the equation

$$f(x) = y$$

and this solution is given by the formula $x = f^{-1}(y)$.