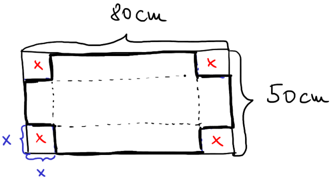
Extrema - practical problems

Matematika (MTL)

LDF MENDELU

Paper box

We would like to make a box from a rectangular sheet of paper. The size of the paper is $80\,\mathrm{cm} \times 50\,\mathrm{cm}$. To obtain the box, we cut a square in each corner of the paper and fold the rectangles. Determine the optimal size of the squares to be cut, such that the obtained box has the maximal volume.



$$V(x) = (80-2x)\cdot(50-2x)\cdot x \longrightarrow \max$$

$$\underline{x \in (0,25)}$$

$$V(x) = 4(40-x)(25-x)\cdot x = 4\cdot(x^3 - 65x^2 + 1000x)$$

$$V'(x) = 4\cdot(3x^2 - 130x + 1000)$$

$$3x^2 - 130x + 1000 = 0$$

$$x_{1,2} = \frac{130 \pm \sqrt{16900 - 12000}}{6} = \frac{130 \pm 70}{6} = \begin{cases} \frac{100}{3} > 25 \text{ not possible} \\ \underline{\underline{10}} \ \checkmark \end{cases}$$

$$V'(x): \quad \overset{+}{0 \nearrow 10} \ \overset{-}{\searrow 25} \quad \frac{100}{3}$$

$\Rightarrow$ We cut squares of the size 10 cm × 10 cm.

The volume of the box is $V = 60\cdot 30\cdot 10 = \underline{18\,000\ \text{cm}^3}$.

Minimal surface area of a cylinder

For the cylinder of a given volume find the ratio of the height to radius such that the cylinder has the smallest surface area.

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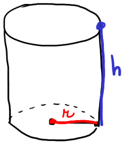

$V=\pi r^2\cdot h$, $S = 2\cdot\pi r^2 + 2\pi r\cdot h$

Let $V=1 \Rightarrow \pi\cdot r^2\cdot h = 1 \Rightarrow h = \frac{1}{\pi r^2}$

$$S(r) = 2\pi r^2 + 2\pi r\cdot\frac{1}{\pi r^2} = 2\pi r^2 + 2\cdot r^{-1}$$

$$\frac{dS}{dr} = 4\pi r - 2\cdot r^{-2} = 4\pi r - \frac{2}{r^2} = \frac{4\pi r^3 - 2}{r^2}$$

$$\frac{dS}{dr}=0 : \quad 4\pi r^3 - 2 = 0 \Rightarrow r^3 = \frac{1}{2\pi} \Rightarrow r = \frac{1}{\sqrt[3]{2\pi}}$$

$$h = \frac{1}{\pi\cdot r^2} = \frac{1}{\pi}\cdot\sqrt[3]{4\pi^2} = \sqrt[3]{\frac{4\pi^2}{\pi^3}} = \sqrt[3]{\frac{4}{\pi}}$$

ratio : $\frac{h}{r} = \frac{\sqrt[3]{4}}{\sqrt[3]{\pi}}\cdot\sqrt[3]{2}\cdot\sqrt[3]{\pi} = \sqrt[3]{8} = 2 \Rightarrow h:r = 2:1$

Fenced area

We would like to fence a rectangular area. One side of the area is a wall, so we need to fence the three sides of the area.

(a) The fencing material is given. Find the dimensions of the rectangular region with the maximal area.

(b) The area of the region is given. Find the dimensions of the rectangular region such that the amount of the fencing material is minimal.

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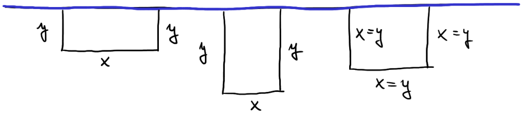


a) $L = x + 2y$ const

$S = x \cdot y \to \max.$

b) $S = x \cdot y$ const

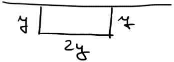
$L = x + 2y \to \min$

a) $L = x + 2y$ const. $\Rightarrow x = L - 2y$

$$S = (L-2y)\cdot y = Ly - 2y^2$$

$$\frac{dS}{dy} = L - 4y = 0 \Rightarrow \underline{\underline{y = \frac{L}{4}}} \Rightarrow \underline{\underline{x}} = L - \frac{L}{2} = \underline{\underline{\frac{L}{2}}}$$

$$\Rightarrow \underline{\underline{x : y = 2 : 1}}$$



b) $S = x \cdot y$ const. $\Rightarrow x = \frac{S}{y}$

$$L = \frac{S}{y} + 2y = S\cdot y^{-1} + 2y$$

$$\frac{dL}{dy} = -S\cdot y^{-2} + 2 = -\frac{S}{y^2} + 2 = 0 \Rightarrow y^2 = \frac{S}{2} \Rightarrow \underline{\underline{y = \sqrt{\frac{S}{2}}}}$$

$$\underline{\underline{x}} = \frac{S}{y} = S\cdot\sqrt{\frac{2}{S}} = \underline{\underline{\sqrt{2}\cdot\sqrt{S}}} \Rightarrow \frac{x}{y} = \sqrt{2}\cdot\sqrt{S}\cdot\frac{\sqrt{2}}{\sqrt{S}} = 2$$

$$\Rightarrow \underline{\underline{x : y = 2 : 1}}$$