Extremal problems and concavity

Mathematics – RRMATA

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Extremal problems

Definition (local extrema)

Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathrm{Dom}(f)$. The function f is said to take on its local maximum at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) \geq f(x)$ for all $x \in N(x_0)$.
The function f is said to take on its sharp local maximum at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) > f(x)$ for $x \in N(x_0) \setminus \{x_0\}$.
If the opposite inequalities hold, then the function f is said to take on its local minimum or sharp local minimum at the point x_0.
A common word for local minimum and maximum is a local extremum (pl. *extrema*). A common word for the sharp local maximum and the sharp local minimum is a sharp local extremum.

Theorem (sufficient conditions for (non-)existence of local extrema)

Let f be a function defined and continuous in some neighborhood of x_0.

- *If the function f is increasing in some left-hand side neighborhood of the point x_0 and decreasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local maximum at the point x_0.*
- *If the function f is decreasing in some left-hand side neighborhood of the point x_0 and increasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local minimum at the point x_0.*
- *If the function f is either increasing or decreasing in some (two-sided) neighborhood of the point x_0, then there is no local extremum of the function f at x_0.*

Definition (stationary point)

The point x_0 is said to be a stationary point of the function f if $f'(x_0) = 0$.

Theorem (relationship between stationary point and local extremum)

Let f be a function defined in x_0. If the function f takes on a local extremum at $x = x_0$, then the derivative of the function f at the point x_0 either does not exist or equals zero and hence $x = x_0$ is a stationary point of the function f.

Theorem (relationship between derivative and monotonicity)

Let f be a function. Suppose that f is differentiable on the open interval I.

- *If $f'(x) > 0$ on I, then the function f is increasing on I.*
- *If $f'(x) < 0$ on I, then the function f is decreasing on I.*

Concavity

Another property of functions which possesses a clear interpretation on the graph is concavity.

Definition (concavity)

Let f be a function differentiable at x_0.
The function f is said to be concave up (concave down) at x_0 if there exists a ring neighborhood $\overline{N}(x_0)$ of the point x_0 such that for all $x \in \overline{N}(x_0)$ the points on the graph of the function f are above (below) the tangent to the graph in the point x_0, i.e. if

$$(1) \qquad f(x) > f(x_0) + f'(x_0)(x - x_0) \qquad \Big(f(x) < f(x_0) + f'(x_0)(x - x_0)\Big)$$

holds.
The function is said to be *concave up (concave down) on the interval I* if it has this property in each point of the interval I.

Definition (point of inflection)

The point x_0 in which the type of concavity changes is said to be a point of inflection of the function f.

Theorem (relationship between the 2nd derivative and concavity)

Let f be a function and f'' be the second derivative of the function f on the open interval I.

- *If $f''(x) > 0$ on I, then the function f is concave up on I.*
- *If $f''(x) < 0$ on I, then the function f is concave down on I.*

Theorem (2nd derivative test, concavity and local extrema)

Let f be a function and x_0 a stationary point of this function.

- *If $f''(x_0) > 0$, then the function f has its local minimum at the point x_0.*
- *If $f''(x_0) < 0$, then the function f has its local maximum at the point x_0.*
- *If $f''(x_0) = 0$, then the 2nd derivative test fails. A local extremum may or may not occur. Both cases are possible.*

Behavior of the function near infinity, asymptotes

In the remaining part of this chapter we will investigate functions near the points $+\infty$ and $-\infty$. We will be interested in the fact, whether the graph approaches a line or not.

Definition (inclined asymptote)

Let f be a function defined in some neighborhood of $+\infty$. The line $y = kx + q$ is said to be an inclined asymptote at $+\infty$ to the graph of the function $y = f(x)$ if

$$\lim_{x \to \infty} |kx + q - f(x)| = 0$$

holds.
Similarly, if we consider the point $-\infty$ instead of $+\infty$, we obtain the definition of the inclined asymptote at $-\infty$.

Remark

As a special case of the preceding definition we obtain for $k = 0$ the horizontal asymptote.

Theorem (inclined asymptote)

Let f be a function defined in some neighborhood of the point $+\infty$. The line $y = kx + q$ is an inclined asymptote at $+\infty$ to the graph of the function $f(x)$ if and only if the following limits exist as finite numbers

(2) $$k := \lim_{x\to\infty} \frac{f(x)}{x} \qquad \textit{and} \qquad q := \lim_{x\to\infty} (f(x) - kx).$$

Similarly, if we consider $-\infty$ instead of $+\infty$, we obtain the asymptote at $-\infty$.

Investigation of a function, curve sketching

Usually we investigate a function f in the following steps.

1. We find the domain of f, decide whether f is odd, even or periodical.
2. We find intercepts of the graph with axes and intervals where the value of the function is positive and/or negative.
3. We find the one-sided limits at the points of discontinuity and at $\pm\infty$.
4. We find and simplify the derivative f'. Then we find intervals where f is increasing and/or decreasing and local extrema of the function f.
5. We find and simplify the second derivative f''. Then we find the intervals where f is concave up and/or down and inflection points of the function f.
6. If the limits in $+\infty$ and/or $-\infty$ are not finite, we find inclined asymptotes in these points.
7. We sketch the graph of the function f.