Derivative

Mathematics – RRMATA

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Derivative

Definition (derivative at a point)

Let f be a function and let $x \in \mathrm{Dom}(f)$. The function f is said to be differentiable at the point x if the finite limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

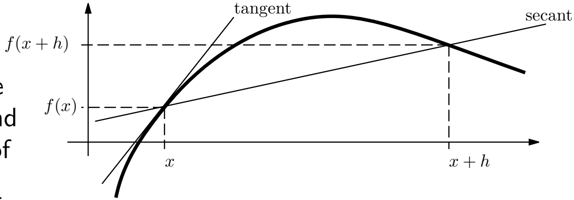
exists. The value of this limit is called a derivative of the function f at the point x.

Definition (derivative as a function)

The function is said to be *differentiable on an open interval* I if it is differentiable at every point of this interval. The function which assigns to each point x from the interval the value $f'(x)$ of the derivative is said to be a *derivative of the function* f *on the interval* I and denoted by f'.

Geometric interpretation of the derivative

Consider a function f and a point x as on the figure. Further consider a secant to this graph which intersects the graph at the points $[x, f(x)]$ and $[x+h, f(x+h)]$. The slope of this line is



$$(2) \qquad \frac{f(x+h) - f(x)}{h}.$$

Now let us (in mind) fix the point $[x, f(x)]$ and move the point $[(x+h), f(x+h)]$ along the graph towards to this fixed point. With the limit process $h \to 0$ these two points become identical and the secant becomes to be a tangent to the graph of the function f at the point x. The slope of this tangent is the limit of the slopes of secants, i.e. the limit of (2). This limit is by definition and by (1) the derivative of the function f at the point x.

Remark (tangent)

If the function f is differentiable at the point a, then the point–slope form of the equation of the tangent line in the point a is

$$y = f'(a)(x - a) + f(a). \tag{3}$$

Remark (practical interpretation of the derivative)

Let the quantity x denotes time (in convenient units) and suppose that the value of the quantity y changes in the time, i.e. $y = y(x)$. Derivative $y'(x)$ of the function y in the point x denotes the instant rate (velocity) of the change of the function y at the time x. As a practical example consider the following situation. Let the quantity y denotes the size of population of some species in some bounded area. In this case the derivative $y'(x)$ denotes the rate of the change of the size of this population. This change equals to the number of the individuals which are born in the moment x decreased by the amount of individuals which died in this moment (more precisely in the time interval which starts at given time and has unit length).

Definition (higher derivatives)

Let $f(x)$ be a function and $f'(x)$ be the derivative of this function. Suppose that there exists derivative $(f'(x))'$ of the function $f'(x)$. Then this derivative is said to be the *second derivative of the function* f and denoted $f''(x)$. By n-times repetition of this process we obtain the n-th derivative $f^{(n)}(x)$ of the function f.

Theorem (relationship between the differentiability and continuity)

Let f be a function differentiable at the point $x = a$ (on the interval I). Then f is continuous at the point $x = a$ (on the interval I).

Notation. The set of all functions with continuous derivative on the interval I is denoted by $C^1(I)$. These functions are called *smooth functions*.

Theorem (algebra of derivatives)

Let f, g be functions and $c \in \mathbb{R}$ be a real constant. The following relations hold

$$
\begin{aligned}
[cf(x)]' &= cf'(x), && \textit{the constant multiple rule} \\
[f(x) \pm g(x)]' &= f'(x) \pm g'(x), && \textit{the sum rule} \\
[f(x)g(x)]' &= f(x)g'(x) + f'(x)g(x), && \textit{the product rule} \\
\left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}, && \textit{the quotient rule}
\end{aligned}
$$

whenever the derivatives on the right-hand side exist and the expression on the right-hand side is well defined.

Theorem (chain rule)

Let f and g be differentiable functions. The relation

$$[f(g(x))]' = f'(g(x))g'(x) \tag{4}$$

holds whenever the right hand side is well defined.

Remark (another notation)

An equivalent notation for the derivative of the function $y = f(x)$ is

$$y'(x) = f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{5}$$

This notation is used especially in applications. In the applications the functions usually contain several constants or parameters and derivative denoted by (5) shows which variable is differentiated and which symbol is considered as an independent variable. The chain rule written in this notation has the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}v}\,\frac{\mathrm{d}v}{\mathrm{d}x}$$

where u is a function of v, v is a function of x, and we differentiated the composite function $u(v(x))$. This rule is easy to remember since the notation is similar to the multiplication of fractions, where the term dv “cancels”.

Formulas for differentiation

$$c' = 0 \qquad (x^n)' = nx^{n-1}$$

$$(e^x)' = e^x \qquad (\sin x)' = \cos x \qquad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(a^x)' = a^x \ln a \qquad (\cos x)' = -\sin x \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\ln x)' = \frac{1}{x} \qquad (\mathrm{tg} x)' = \frac{1}{\cos^2 x} \qquad (\mathrm{arctg} x)' = \frac{1}{1+x^2}$$

$$(\log_a x)' = \frac{1}{x \ln a} \qquad (\mathrm{cotg} x)' = -\frac{1}{\sin^2 x} \qquad (\mathrm{arccotg} x)' = -\frac{1}{1+x^2}$$

Example

1. $y = \sin x(x^2 + 3x)$

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Linear approximation of a function

Let f be a function differentiable at the point $x = a$. Then we can find the equation of its tangent in the point a by using (3). From the graph is clear that this tangent is the best linear function which approximates $f(x)$ near the point a. Hence we can write an approximate formula

$$f(x) \approx f(a) + f'(a)(x - a) \tag{6}$$

which approximates the function f by a linear function. Remember that this approximation is usually convenient for the points very close to the point a only. If this linear approximation is not sufficient in a particular problem, we can approximate the function f by a higher degree polynomial (Taylor formula).

Newton–Raphson method

The Newton–Raphson method (like bisection method) is another method for approximation of the zeros of functions. Suppose that we have to solve $f(x) = 0$ and x_0 is the initial estimate for this solution. We write tangent to the graph of the funciton $f(x)$ at $x = x_1$

$$y = f'(x_1)(x - x_1) + f(x_1)$$

and find zero x_2 of this tangent:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now x_2 is improved approximation for zero of $f(x)$. If the zero fo the fucntion $f(x)$ is $x = c$, $f'(c)$ is not zero and x_1 is sufficiently close to c, then the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to c.

Partial derivative

Definition (Partial derivative)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of 2 variables which is defined at the point (x_0, y_0) and in a neighborhood of this point.
We define the partial derivatives of the function f at the point (x_0, y_0) as follows.

- partial derivative of f with respect to x at (x_0, y_0):

$$f'_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

- partial derivative of f with respect to y at (x_0, y_0):

$$f'_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Partial derivative as a function

Let the function $z = f(x, y)$ has partial derivative with respect to x at all points of the set $M \subset \mathrm{Dom}(f)$. Then it is possible to define a function that associates the partial derivative with respect to x to each point $x \in M$. This function is called partial derivative of the functioin f with respect to x and is denoted f'_x. In the same way we define the partial derivative with respect to y and denote f'_y.

Another notation of partial derivatives:

$$f_x,\ f_y, \quad \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}, \quad z'_x,\ z'_y, \quad z_x,\ z_y, \quad \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y}$$

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1. $z = x^3 + 2x^2y^2 + 3x^2y - 6xy + 8x - 2$

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$$z'_x = yx^{y-1} \quad \text{(power function)}$$

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$$\begin{aligned} z'_x &= yx^{y-1} \quad \text{(power function)} \\ z'_y &= x^y \ln x \quad \text{(exponencial function)} \end{aligned}$$

Second order partial derivatives

Definition (Second order partial derivatives)

The partial derivatives of the functions f'_x, f'_y are called the second order partial derivatives of the function f.

Notation:

- partial derivative of f'_x with respect to x: f''_{xx}, or $\frac{\partial^2 f}{\partial x^2}$
- partial derivative of f'_x with respect to y: f''_{xy}, or $\frac{\partial^2 f}{\partial xy}$
- partial derivative of f'_y with respect to x: f''_{yx}, or $\frac{\partial^2 f}{\partial yx}$
- partial derivative of f'_y with respect to y: f''_{yy}, or $\frac{\partial^2 f}{\partial y^2}$

Example

Find the second order partial derivatives of the function $z = e^{x^2 y}$.

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z''_{xx} &= 2y \cdot e^{x^2 y} + 2xy \cdot e^{x^2 y} \cdot 2xy = 2ye^{x^2 y}(1 + 2x^2 y) \\
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\end{aligned}
$$

Example

Find the second order partial derivatives of the function $z = e^{x^2 y}$.

$$
\begin{aligned}
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We can see that $z''_{xy} = z''_{yx}$. This holds when z''_{xy}, z''_{yx} are continuous.

Using the computer algebra systems

Wolfram Alpha:

`http://www.wolframalpha.com/`

Example

Differentiate the following functions:

$$f(x) = \sin(x^2 + 1), \quad g(x) = \frac{x+2}{x-1}.$$

Wolfram Alpha:

```
differentiate sin(x^2+1)

differentiate (x+2)/(x-1)
```

Example

Find the partial derivatives of the function

$$z = x^2 \ln(x + y^3).$$

Wolfram Alpha:

- `differentiate x^2*ln(x+y^3) with respect to x`
- `differentiate x^2*ln(x+y^3) with respect to y`