INVESTMENTS IN EDUCATION DEVELOPMENT

Calculus

Mathematics – FRDIS

MENDELU

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Functions

Definition (function)

Let A and B be nonempty sets of real numbers.
Let f be a rule which associates each element x of the set A with exactly one element y of the set B. The rule f is said to be a *function* defined on A. We write $\boldsymbol{f : A \to B}$. If f associates x with y, we write $\boldsymbol{y = f(x)}$.
The variable x is customary called an *independent variable* and y a *dependent variable*.
The set A is called a *domain* of the function f and denoted by $\mathbf{Dom}(\boldsymbol{f})$.
The set B is a *target set*. The subset of all that elements y of the set B which are generated by the elements from $\mathrm{Dom}(f)$ is called an *image* (or *range*) of the function f and denoted by $\mathbf{Im}(\boldsymbol{f})$.

Definition (graph)

Let f be a function.
A graph of the function f is the set of all of the points in the plane $[x, y] \in \mathbb{R}^2$ with property $y = f(x)$.
If $0 \in \mathrm{Dom}(f)$, then the value $f(0)$ is well-defined and the point point $[0, f(0)]$ on the graph is called an y-*intercept* of the graph.
The number $x \in \mathrm{Dom}(f)$ which satisfies $f(x) = 0$ is called a *root* or a *zero* of the function f.
If the number x_0 is a zero of the function f, then the point $[x_0, 0]$ on the graph is called an x-*intercept* of the graph.

Definition (composite function)

Let $f : A \to B$ and $g : B \to C$ be functions. Under a *composite function* $g \circ f$ (read "g of f") we understand the function defined for every $x \in A$ by the relation

$$(g \circ f)(x) = g(f(x)).$$

The function f is said to be an *inside function* and g an *outside function* of the composite function $g \circ f$.

Example

$f : y = x^2, \quad g : y = \sin x$

- $g \circ f : y = g(f(x)) = \sin x^2$
- $f \circ g : y = f(g(x)) = \sin^2 x$

The composite function may consist of more than two functions, e.g.:

$$(h \circ g \circ f)(x) = h(g(f(x))).$$

Example

$f : y = x^2, \quad g : y = \sin x, \quad h : y = \ln x$

- $h \circ g \circ f : y = h(g(f(x))) = \ln \sin x^2$
- $f \circ g \circ h : y = f(g(h(x))) = \sin^2 \ln x$
- $f \circ h \circ g : y = f(h(g(x))) = \ln^2 \sin x$
- $g \circ f \circ h : y = g(f(h(x))) = \sin \ln^2 x$

Remark (basic elementary functions, elementary functions)

Several functions which are common in applications have special names. This concerns especially the following functions:

1. *Power function* $y = x^\alpha$, $\alpha \in \mathbb{R}$
2. *Exponential function* $y = a^x$, $a \in \mathbb{R}^+ \setminus \{1\}$
3. *Logarithmic function* $y = \log_a x$, $a \in \mathbb{R}^+ \setminus \{1\}$
4. *Trigonometric functions* $y = \sin x$, $y = \cos x$, $y = \mathrm{tg} x$, $y = \mathrm{cotg} x$
5. *Inverse trigonometric functions* $y = \arcsin x$, $y = \arccos x$, $y = \mathrm{arctg} x$, $y = \mathrm{arccotg} x$

The functions in the preceeding list are called *basic elementary functions*.
A sum of constant multiples of power functions with a nonnegative integer exponent is called a *polynomial*, e.g. $y = 2x^3 - 4x + 1$ is a polynomial. The highest power in the polynomial is called a *degree* of the polynomial.
The quotient of two polynomials is called a *rational function*, e.g. $y = \frac{x}{x^3 - 2x + 1}$ is a rational function.

Remark (basic elementary functions, elementary functions)

The function f which can be represented by a single formula $y = f(x)$, where the expression on the right-hand side is made up of basic elementary functions and constants by means of a finite number of the operations addition, subtraction, multiplication, division and composition is called an *elementary function*. In the exercises we will almost exclusively work with elementary functions, since this is sufficient to describe most of the phenomena in scientific and economical applications.

Remark

In calculus we are interested especially in the composition of the basic elementary functions. Hence $y = \ln(x+1)$, $y = \arcsin\sqrt{x}$ and $y = \sin\frac{x}{2}$ are treated as composite functions. On the other hand, the basic elementary functions like $y = e^x$ and $y = \mathrm{tg} x$ are not composite functions.

Basic properties of functions

Definition (odd and even function)

Let f be a function with domain $\mathrm{Dom}(f)$. Let for every $x \in \mathrm{Dom}(f)$ also $-x \in \mathrm{Dom}(f)$.

1. The function f is said to be *even*, if $f(-x) = f(x)$ holds for every $x \in \mathrm{Dom}(f)$.
2. The function f is said to be *odd*, if $f(-x) = -f(x)$ holds for every $x \in \mathrm{Dom}(f)$.

Arrangement: The function is said to have a parity if it is either odd or even.

Definition (boundedness)

Let f be a function and $M \subseteq \mathrm{Dom}(f)$ be a subset of the domain of the function f.

1. The function f is said to be *bounded from below* on the set M if there exists a real number a with the property $a \leq f(x)$ for all $x \in M$.
2. The function f is said to be *bounded from above* on the set M if there exists a real number a with the property $a \geq f(x)$ for all $x \in M$.
3. The function f is said to be *bounded* on the set M if it is bounded from both below and above on the set M.

If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

Motivation. In the following paragraphs we will be interested in the fact, whether an application of the function f on both sides of an equality yields an equivalent equality. We start with an obvious implication which holds for every well-defined function f.

$$x_1 = x_2 \implies f(x_1) = f(x_2) \tag{1}$$

The equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent, if the implication in (1) can be reversed. It is clear that this implication *cannot* be reversed if two mutually different values $x_1 \neq x_2$ in the domain of f may generate the same value of the function f. A property which excludes this possibility is introduced in the following definition.

Definition (one-to-one function)

Let f be a function and let $M \subseteq \mathrm{Dom}(f)$ be a subset of the domain of the function f.
The function f is said to be *one-to-one function* on the set M if there exists no pair of different elements x_1, x_2 of the set M which are associated with the same element y of the image $\mathrm{Im}(f)$.
If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

Remark (one-to-one functions)

Mathematically formulated, for the one-to-one functions the following implication holds

(2) $$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This implication means that if the function f is one-to-one, then we can remove (or more precisely un-apply) this function from both sides of the relation $f(x_1) = f(x_2)$ and conclude the equivalent relation $x_1 = x_2$. This property can be used when solving nonlinear equations, since this is an exact description of the fact that the equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent.

Definition (inverse function)

Let $f : A \to B$ be an one-to-one function. The rule which associates every x with the number y satisfying $f(y) = x$ defines a function on B (really, for every $x \in \mathrm{Im}(f) \subseteq B$ there exists only one y satisfying $f(y) = x$). This function is said to be an *inverse function* to the function f. The inverse function to the function f is denoted by $\boldsymbol{f^{-1}}$.

Remark

The inverse function to some of the basic elementary function is sometimes simply a name of another elementary function. Table 1 summarizes the basic elementary function and their inverses. The property to be an inverse function is mutual. If, for example, the $\arcsin(\cdot)$ function is inverse to the $\sin(\cdot)$ function, then also the $\sin(\cdot)$ function is an inverse to the $\arcsin(\cdot)$ function.

The function $y = f(x)$	The inverse function $y = f^{-1}(x)$
$y = \sqrt{x}$	$y = x^2,\ x > 0$
$y = x^2,\ x > 0$	$y = \sqrt{x}$
$y = e^x$	$y = \ln x$
$y = \ln x$	$y = e^x$
$y = a^x$	$y = \log_a x$
$y = \sin x,\ x \in [-\pi/2, \pi/2]$	$y = \arcsin x$
$y = \cos x,\ x \in [0, \pi]$	$y = \arccos x$
$y = \mathrm{tg} x,\ x \in [-\pi/2, \pi/2]$	$y = \mathrm{arctg} x$

Tabulka: Inverse functions to the basic elementary functions

Remark (nonlinear equations)

The inverse function allows an alternative interpretation in terms of solutions of nonlinear equations: If there exists an inverse f^{-1} of the function f and if the inverse function is defined in some x, then the nonlinear equation with the unknown y

$$f(y) = x \tag{3}$$

has exactly one solution. This solution is given by the formula

$$y = f^{-1}(x). \tag{4}$$

An alternative method how to obtain (4) from (3) is to write the right-hand side of (3) in the form $f(f^{-1}(x))$ and conclude (4) via implication (2).

Motivation. Now we will distinguish the classes of functions which preserve or reverse inequalities. This yields the following classes of functions.

Definition (monotonicity)

Let f be a function and $M \subseteq \mathrm{Dom}(f)$ be a subset of its domain.

1. The function f is said to be *increasing* on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) < f(x_2)$ holds.
2. The function f is said to be *decreasing* on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) > f(x_2)$ holds.
3. The function f is said to be *(strictly) monotone* on the set M if it is either increasing or decreasing on M.

If the set M is not specified, we suppose $M = \mathrm{Dom}(f)$.

Remark (summing up remark)

Finally, we summarize all the properties of the functions introduced in this chapter. We use the notation which is usefull when solving equations and (or) inequalities.

$$a = b \overset{f \text{ is one-to-one}}{\iff} f(a) = f(b)$$

$$a < b \overset{f \text{ is increasing}}{\iff} f(a) < f(b) \qquad a \leq b \overset{f \text{ is increasing}}{\iff} f(a) \leq f(b)$$

$$a < b \overset{f \text{ is decreasing}}{\iff} f(a) > f(b) \qquad a \leq b \overset{f \text{ is decreasing}}{\iff} f(a) \geq f(b)$$

If the function f is one-to-one, then for every $y \in \mathrm{Im}(f)$ there exists a unique solution x of the equation

$$f(x) = y$$

and this solution is given by the formula $x = f^{-1}(y)$.

Limit, continuity

Definition (expanded set of real numbers)

Under *an expanded set of real numbers* $\mathbb{R}^*$ we understand the set $\mathbb{R}$ of all real numbers enriched by the numbers $\pm\infty$ in the following way: We set $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ and for $a \in \mathbb{R}$ we set:

$$a + \infty = \infty, \qquad a - \infty = -\infty, \qquad \infty + \infty = \infty, \qquad -\infty - \infty = -\infty$$

$$\infty \cdot \infty = -\infty \cdot (-\infty) = \infty, \qquad \infty \cdot (-\infty) = -\infty, \qquad \frac{a}{\infty} = \frac{a}{-\infty} = 0$$

$$-\infty < a < \infty, \qquad |\pm\infty| = \infty.$$

Further, for $a > 0$ we set

$$a \cdot \infty = \infty \qquad a \cdot (-\infty) = -\infty,$$

and for $a < 0$ we set

$$a \cdot \infty = -\infty \qquad a \cdot (-\infty) = \infty.$$

Another operations we define with the commutativity of the operation "$+$" and "$\cdot$".

Remark (indeterminate forms)

The operations "$\infty - \infty$", "$\pm\infty \cdot 0$" and "$\frac{\pm\infty}{\pm\infty}$" remain undefined. Of course, the division by a zero remains undefined as well.

Definition (neighborhood)

Under the *neighborhood* of the point $a \in \mathbb{R}$ we understand any open interval which cotains the point a, we write $\boldsymbol{N(a)}$. Under the *reduced* (also *ring*) *neighborhood* of the point a we understand the set $N(a) \setminus \{a\}$, we write $\boldsymbol{\overline{N}(a)}$. Under the *neighborhood of the point* ∞ we understand the interval of the type (A, ∞) and under the *neighborhood of the point* $-\infty$ the interval $(-\infty, A)$. Under the reduced neighborhood of the points $\pm\infty$ we understand the same as under the neighborhood of these points.

Definition (limit of the function)

Let $a, L \in \mathbb{R}^*$ and $f : \mathbb{R} \to \mathbb{R}$. Let the function f be defined in some reduced neighborhood of the point a. We say that the function $y = f(x)$ approaches to the *limit* L as x approaches to a if for any (arbitrary small) neighborhood $N(L)$ of the number L there exists reduced neighborhood $\overline{N}(a)$ of the point a such that for every $x \in \overline{N}(a)$ the relation $f(x) \in N(L)$ holds.
We write

$$\lim_{x \to a} f(x) = L \tag{5}$$

or $\boldsymbol{f(x) \to L}$ for $\boldsymbol{x \to a}$.

Arrangement. Shortly we read (5) as "the limit of f at a is L".

Motivation. From a geometric point of view it turns out to be interesting to distinguish the cases in which x approaches to a from the left and from the right. This gives a motivation for the following definitions.

Definition (one-sided neighborhood)

Under the *right (left) neighborhood* of the point $a \in \mathbb{R}$ we understand the interval of the type $[a, b)$,(or $(b, a]$, for left neighborhood), we write $N^+(a)$ $(N^-(a))$. Under the *reduced right (left) neighborhood* of the point a we understand the corresponding neighborhood wihtout the point a, we write $\overline{N^+}(a)$, $(\overline{N^-}(a))$

Definition (one-sided limit)

If we replace in the definition of the limit the reduced neighborhood of the point a by the reduced right neighborhood of the point a, we obtain a definition of the *limit from the right*. We write $\lim_{x\to a^+} f(x) = L$.
Similarly, we define also the limit from the left. In this case we write $\lim_{x\to a^-} f(x) = L$.

Remark (shortened notation)

Another (very short) notation for one-sided limits is $f(a+) = L$ for the limit from the right and $f(a-) = L$ for the limit from the left. For the two-sided limit we can write $f(a\pm)$. In several textbooks also the notation $f(a + 0)$, $f(a - 0)$ and $f(a \pm 0)$, respectively, is used.

Theorem (uniqueness of the limit)

The function f possesses at the point a at most one limit (or one-sided limit).

Theorem (the relationship between the limit and the one-sided limits)

The limit of the function f at the point $a \in \mathbb{R}$ exists if and only if both one-sided limits at the point a exist and are equal. More precisely: If the limits $f(a-)$ and $f(a+)$ exist and $f(a-) = f(a+)$, then the limit $f(a\pm)$ exists as well and $f(a\pm) = f(a-) = f(a+)$. If one of the one-sided limits does not exist or if $f(a-) \neq f(a+)$, then the limit $f(a\pm)$ does not exist.

Definition (continuity at a point)

Let f be a function defined at the point $a \in \mathbb{R}$.
The function f is said to be *continuous* at the point a if $\lim_{x\to a} f(x) = f(a)$.
The function f is said to be *continuous from the right* at the point a if $\lim_{x\to a^+} f(x) = f(a)$.
The function f is said to be *continuous from the left* at the point a if $\lim_{x\to a^-} f(x) = f(a)$.

Remark

According to the definition, the function is continuous at the point a if

- $f(a)$ exists,
- $\lim_{x\to a} f(x)$ exists as finite number,
- $f(a) = \lim_{x\to a} f(x)$ holds.

If the function is defined in some (at least one-sided) reduced neighborhood of the point a but at least one of the three conditions above is broken, the point $x = a$ is said to be a point of *discontinuity* of the function f.

Definition (continuity on an interval)

The function is said to be *continuous on the open interval* (a, b) if it is continuous at every point of this interval.
The function is said to be *continuous on the closed interval* $[a, b]$ if it is continuous on (a, b), continuous from the right at the point a and continuous from the left at the point b.

Notation. The class of all functions continuous on the interval I will be denoted by $C(I)$. If $I = (a, b)$ or $I = [a, b]$, then we write shortly $C((a, b))$ or $C([a, b])$, respectively.

Theorem (continuity of elementary functions)

Every elementary function is continuous on its domain.

Remark (practical)

Consider an elementary function $f(x)$ and the limit of this function for $x \to a$. According to the preceding theorem, we try to substitute a for x into $f(x)$ and calculate $f(a)$ first. If this is possible, i.e. if the number a is in the domain of f, then we obtain the limit. If this fails, i.e. if $f(a)$ is not defined, we must look for another method. Hence, when speaking about elementary functions, the concept of limit gives *nothing new concerning the points from the domain*. This concept remains interesting only for the points, which are not in the natural domain of the function, but the function is defined in some ring neighborhood of these points.

Definition (vertical asymptote)

Let f be a function and x_0 a real number. The vertical line $x = x_0$ is said to be a *vertical asymptote to the graph of the function* f if at lest one of the one-sided limits of the function f at the point x_0 exists and it is not a finite number.

Definition (horizontal asymptote)

The line $y = L$ is said to be a *horizontal asymptote to the graph of the function* $f(x)$ *at* $+\infty$ if the limit of the function f at $+\infty$ exists and

$$\lim_{x\to\infty} f(x) = L$$

holds. In a similar way we define the *horizontal asymptote at* $-\infty$.

Theorem (algebra of limits)

Let $a \in \mathbb{R}^*$, $f, g : \mathbb{R} \to \mathbb{R}$. *The following relations hold whenever the limits on the right exist and the formula on the right is well-defined.*

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \tag{6}$$

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \tag{7}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{8}$$

The same holds for one sided limits as well.

Theorem (limit of the composite function with continuous component)

Let $\lim_{x\to a} f(x) = b$ *and* $g(x)$ *be a function continuous at* b*. Then* $\lim_{x\to a} g(f(x)) = g(b)$*, i.e.*

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)).$$

The same holds for one-sided limits as well.

Theorem

Let $a \in \mathbb{R}^*$, $\lim_{x\to a} g(x) = 0$ *and* $\lim_{x\to a} f(x) = L \in \mathbb{R}^* \setminus \{0\}$*. Suppose that there exists a ring neighborhood of the point* a *such that the function* $g(x)$ *does not change its sign in this neighborhood. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} +\infty & \text{if } g(x) \text{ and } L \text{ have common sign,} \\ -\infty & \text{if } g(x) \text{ and } L \text{ have an opposite sign,} \end{cases}$$

in the neighborhood under consideration. The same holds for one sided limits as well.

Theorem (limit of the polynomial or of the rational functions at $\pm\infty$**)**

It holds

$$\lim_{x\to\pm\infty} (a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n) = \lim_{x\to\pm\infty} a_0x^n,$$

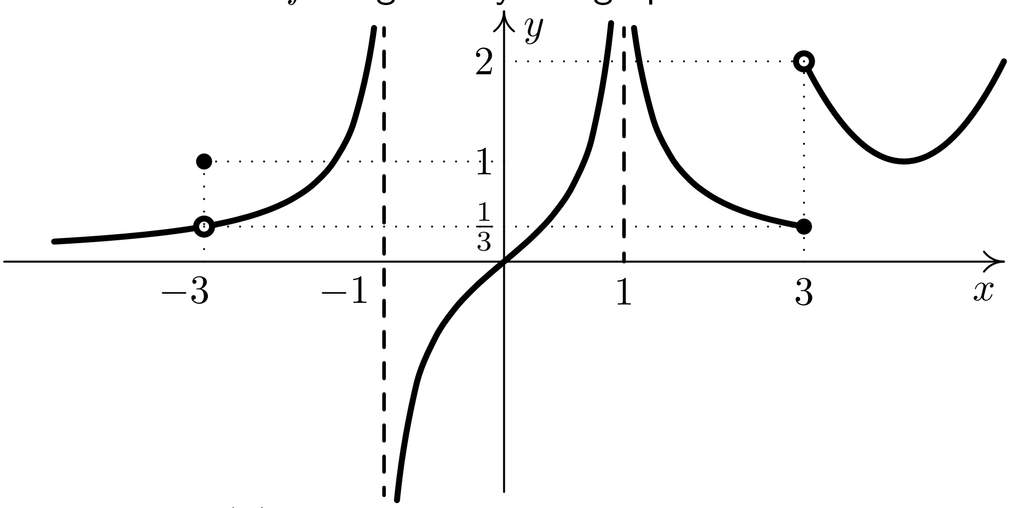
$$\lim_{x\to\pm\infty} \frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} = \lim_{x\to\pm\infty} \frac{a_0}{b_0}x^{n-m}.$$

Remark

The preceding theorem is applicable also in the cases when the rules for algebra of limits give undefined expression $\infty - \infty$ for polynomials or $\frac{\infty}{\infty}$ for rational functions!

Example

Let the function f be given by the graph:



- $\lim\limits_{x\to-\infty} f(x) = 0$
- $\lim\limits_{x\to-3} f(x) = \frac{1}{3}$
- $\lim\limits_{x\to-1} f(x)$ does not exist, since
  $\lim\limits_{x\to-1^-} f(x) = \infty$, $\lim\limits_{x\to-1^+} f(x) = -\infty$
- $\lim\limits_{x\to 1} f(x) = \infty$
- $\lim\limits_{x\to 3} f(x)$ does not exist, since
  $\lim\limits_{x\to 3^-} f(x) = \frac{1}{3}$, $\lim\limits_{x\to 3^+} f(x) = 2$
- $\lim\limits_{x\to\infty} f(x) = \infty$

asymptotes: $x = -1$, $x = 1$ and $y = 0$ for $x \to -\infty$

Example (Limit of the type $\left\| \frac{a}{\pm\infty} \right\|$, $a \in \mathbb{R}$)

1. $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = \dfrac{1}{\infty} = 0$
2. $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\ln x} = \dfrac{0}{-\infty} = 0$

Example (Limit of the type $\left\|\frac{a}{0}\right\|$, $a \in \mathbb{R}^* \setminus \{0\}$)

1. $\lim\limits_{x\to 3} \dfrac{x}{(x-3)^3} = \left\|\dfrac{3}{0}\right\| \Rightarrow$ Possible results: ∞, $-\infty$ or does not exist.
   The function $\frac{x}{(x-3)^3}$ is positive in the right neighborhood of $x = 3$, but it is negative in the left neighborhood of $x = 3$. We have:
   $$\lim_{x\to 3^+} \frac{x}{(x-3)^3} = \left\|\frac{+}{+}\right\| = \infty, \quad \lim_{x\to 3^-} \frac{x}{(x-3)^3} = \left\|\frac{+}{-}\right\| = -\infty,$$
   $\Rightarrow \lim\limits_{x\to 3} \dfrac{x}{(x-3)^3}$ does not exist.
2. $\lim\limits_{x\to 5} \dfrac{x-7}{(x-5)^2} = \left\|\dfrac{-2}{0}\right\| \Rightarrow$ possible results: ∞, $-\infty$ or does not exist.
   The function $\frac{x-7}{(x-5)^5}$ is negative in a small neighborhood of $x = 5$, hence
   $$\lim_{x\to 5} \frac{x-7}{(x-5)^2} = \left\|\frac{-}{+}\right\| = -\infty.$$

Example (Limit of the polynomial and the rational function)

1. $\lim\limits_{x\to -\infty} (2x^3 - 4x^2 + x - 2) = \lim\limits_{x\to -\infty} 2x^3 = -\infty.$
2. $\lim\limits_{x\to \infty} \dfrac{x^4 + 3x^2 - 2x}{3x^4 + 5} = \lim\limits_{x\to \infty} \dfrac{x^4}{3x^4} = \dfrac{1}{3}.$
3. $\lim\limits_{x\to -\infty} \dfrac{2x^6 + 3x^4 - 2}{3x^2 + x - 1} = \lim\limits_{x\to -\infty} \dfrac{2x^6}{3x^2} = \lim\limits_{x\to -\infty} \dfrac{2}{3}x^4 = \infty.$
4. $\lim\limits_{x\to \infty} \dfrac{5x^2 - 3x - 2}{3x^3 + x^2 + 2} = \lim\limits_{x\to \infty} \dfrac{5x^2}{3x^3} = \lim\limits_{x\to -\infty} \dfrac{5}{3x} = \dfrac{5}{-\infty} = 0.$

Theorems on continuous functions

Theorem (Weierstrass)

Let f be a function defined and continuous on $[a, b]$. Then the function f is bounded and takes on an absolute maximum and an absolute minimum on the interval $[a, b]$, i.e. there exist numbers $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

Theorem (Bolzano, the 1st Bolzano's theorem)

Let f be a function defined and continuous on $[a, b]$. If $f(a).f(b) < 0$ holds (i.e. the values $f(a)$ and $f(b)$ have different signs), then there exists a zero of the function f on the interval (a, b), i.e. there exists $c \in (a, b)$ such that $f(c) = 0$.

Theorem (Bolzano, the 2nd Bolzano's theorem)

Let f be a function defined and continuous on $[a, b]$. Let m and M be absolute minimum and absolute maximum of the funciton f on the interval $[a, b]$, respectively. Then for every y_0 between m and M there exists at least one x_0 with property $f(x_0) = y_0$.

Bisection method

The bisection method is a simple method for approximation of the zeros of continuous functions. Given a continuous function $f(x)$ and real numbers $a, b \in \mathbb{R}$, suppose that $f(a)f(b) < 0$ holds. According to the first Bolzano theorem, there exists $\tilde{c} \in [a, b]$ such that $f(\tilde{c}) = 0$. Consider the point $c = \frac{b+a}{2}$ and the value $f(c)$. One of the relations

$$f(a)f(c) < 0 \quad \text{or} \quad f(c)f(b) < 0 \quad \text{or} \quad f(c) = 0$$

holds. Omitting the last possibility (in this case $x = c$ is an exact zero of the function $f(x)$), we see that one of the intervals (a, c) and (b, c) contains a zero of the function $f(x)$ When seeking a zero, we can focus our attention to the appropriate left half or right half of the interval (a, b). Hence the localisation of the zero is somewhat better: the length of the interval with change of sign is one half to the original length and hence the accuracy is two times better. Following this idea we can, after a finite number of steps, obtain the value of the zero with an arbitrary precision.

Derivative

Definition (derivative at a point)

Let f be a function and let $x \in \mathrm{Dom}(f)$. The function f is said to be *differentiable at the point* x if the finite limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

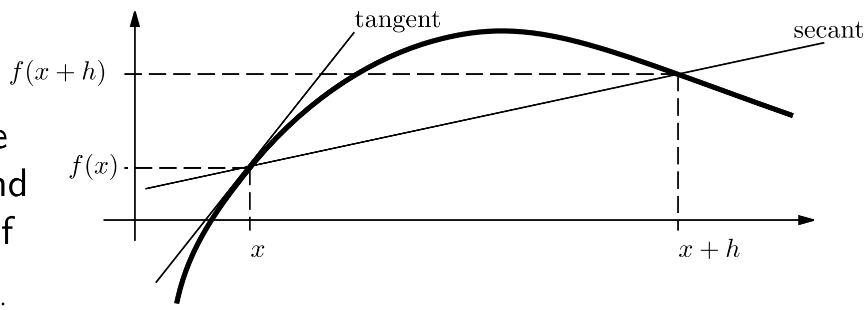
exists. The value of this limit is called a *derivative of the function* f *at the point* x.

Definition (derivative as a function)

The function is said to be *differentiable on an open interval* I if it is differentiable at every point of this interval. The function which assigns to each point x from the interval the value $f'(x)$ of the derivative is said to be a *derivative of the function* f *on the interval* I and denoted by $\boldsymbol{f'}$.

Geometric interpretation of the derivative

Consider a function f and a point x as on the figure. Further consider a secant to this graph which intersects the graph at the points $[x, f(x)]$ and $[x+h, f(x+h)]$. The slope of this line is



$$\frac{f(x+h) - f(x)}{h}. \tag{10}$$

Now let us (in mind) fix the point $[x, f(x)]$ and move the point $[(x+h), f(x+h)]$ along the graph towards to this fixed point. With the limit process $h \to 0$ these two points become identical and the secant becomes to be a tangent to the graph of the function f at the point x. The slope of this tangent is the limit of the slopes of secants, i.e. the limit of (10). This limit is by definition and by (9) the derivative of the function f at the point x.

Remark (tangent)

If the function f is differentiable at the point a, then the point–slope form of the equation of the tangent line in the point a is

$$y = f'(a)(x - a) + f(a). \tag{11}$$

Remark (practical interpretation of the derivative)

Let the quantity x denotes time (in convenient units) and suppose that the value of the quantity y changes in the time, i.e. $y = y(x)$. Derivative $y'(x)$ of the function y in the point x denotes the instant rate (velocity) of the change of the function y at the time x. As a practical example consider the following situation. Let the quantity y denotes the size of population of some species in some bounded area. In this case the derivative $y'(x)$ denotes the rate of the change of the size of this population. This change equals to the number of the individuals which are born in the moment x decreased by the amount of individuals which died in this moment (more precisely in the time interval which starts at given time and has unit length).

Definition (higher derivatives)

Let $f(x)$ be a function and $f'(x)$ be the derivative of this function. Suppose that there exists derivative $(f'(x))'$ of the function $f'(x)$. Then this derivative is said to be the *second derivative of the function* f and denoted $\boldsymbol{f''(x)}$. By n-times repetition of this process we obtain the n-th derivative $\boldsymbol{f^{(n)}(x)}$ of the function f.

Theorem (relationship between the differentiability and continuity)

Let f be a function differentiable at the point $x = a$ (on the interval I). Then f is continuous at the point $x = a$ (on the interval I).

Notation. The set of all functions with continuous derivative on the interval I is denoted by $C^1(I)$. These functions are called *smooth functions*.

Theorem (algebra of derivatives)

Let f, g be functions and $c \in \mathbb{R}$ be a real constant. The following relations hold

$$\begin{aligned}
[cf(x)]' &= cf'(x), && \textit{the constant multiple rule}\\
[f(x) \pm g(x)]' &= f'(x) \pm g'(x), && \textit{the sum rule}\\
[f(x)g(x)]' &= f(x)g'(x) + f'(x)g(x), && \textit{the product rule}\\
\left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}, && \textit{the quotient rule}
\end{aligned}$$

whenever the derivatives on the right-hand side exist and the expression on the right-hand side is well defined.

Theorem (chain rule)

Let f and g be differentiable functions. The relation

(12) $$[f(g(x))]' = f'(g(x))g'(x)$$

holds whenever the right hand side is well defined.

Remark (another notation)

An equivalent notation for the derivative of the function $y = f(x)$ is

(13) $$y'(x) = f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

This notation is used especially in applications. In the applications the functions usually contain several constants or parameters and derivative denoted by (13) shows which variable is differentiated and which symbol is considered as an independent variable. The chain rule written in this notation has the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}$$

where u is a function of v, v is a function of x, and we differentiated the composite function $u(v(x))$. This rule is easy to remember since the notation is similar to the multiplication of fractions, where the term $\mathrm{d}v$ "cancels".

Example

1. $y = \sin x(x^2 + 3x) \;\Rightarrow\; y' = \cos x(x^2 + 3x) + \sin x(2x + 3)$

2. $y = \dfrac{x^3}{x^2 + 1} \;\Rightarrow\; y' = \dfrac{3x^2(x^2 + 1) - x^3 \cdot 2x}{(x^2 + 1)^2} = \dfrac{x^4 + 3x^2}{(x^2 + 1)^2} = \dfrac{x^2(x^2 + 3)}{(x^2 + 1)^2}$

3. $y = \sin x^2 \;\Rightarrow\; y' = \cos x^2 \cdot 2x$

4. $y = \sin^2 x \;\Rightarrow\; y' = 2 \sin x \cdot \cos x$

5. $y = \ln \sin e^{2x} \;\Rightarrow\; y' = \dfrac{1}{\sin e^{2x}} \cdot \cos e^{2x} \cdot e^{2x} \cdot 2$

Linear approximation of a function

Let f be a function differentiable at the point $x = a$. Then we can find the equation of its tangent in the point a by using (11). From the graph is clear that this tangent is the best linear function which approximates $f(x)$ near the point a. Hence we can write an approximate formula

$$f(x) \approx f(a) + f'(a)(x - a) \tag{14}$$

which approximates the function f by a linear function. Remember that this approximation is usually convenient for the points very close to the point a only. If this linear approximation is not sufficient in a particular problem, we can approximate the function f by a higher degree polynomial, as will be explained later (Taylor formula on page 43).

Newton–Raphson method

The Newton–Raphson method (like bisection method) is another method for approximation of the zeros of functions. Suppose that we have to solve $f(x) = 0$ and x_0 is the initial estimate for this solution. We write tangent to the graph of the funciton $f(x)$ at $x = x_1$

$$y = f'(x_1)(x - x_1) + f(x_1)$$

and find zero x_2 of this tangent:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now x_2 is improved approximation for zero of $f(x)$. If the zero fo the fucntion $f(x)$ is $x = c$, $f'(c)$ is not zero and x_1 is sufficiently close to c, then the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to c.

Taylor polynomial

Let f be a real function with the following properties.

- The value $f(x_0)$ is known.
- We have no effective method to evaluate the function at the other points, different from x_0.
- The value of the first n derivatives of the function f at the point x_0 is known.

We state the following problem: *Find an n-degree polynomial which approximates the function f in the neighbourhood of the point x_0 with the smallest possible error.*

The solution of this problem is introduced in the following definition.

Definition (Taylor polynomial)

Let $n \in \mathbb{N}$ be a positive integer and f be a function defined at x_0 which has derivatives up to the order n at x_0. The polynomial

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0}{n!}(x - x_0)^n$$

is called an *n-degree Taylor polynomial of the function f at x_0.*

The point x_0 is called a *centre* of this polynomial.

Theorem (Taylor)

Let f be a function defined in a neighborhood $N(x_0)$ of the point x_0. Let $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0)$ be the values of the first n derivatives of the function f at the point x_0. Suppose that the $(n+1)$st derivative is continuous in $N(x_0)$. Then for all $x \in N(x_0)$

$$f(x) = T_n(x) + R_{n+1}(x),$$

holds, where $T_n(x)$ is the n-degree Taylor polynomial of the function f in the point x_0 and $R_{n+1}(x)$ a remainder. The remainder can be written in the form

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \tag{15}$$

where c is some number between x and x_0.

Remark (polynomial approximation)

From the formula (15) it follows that the remainder is small if

- $(x - x_0)$ is small, i.e., x is close to x_0,
- $n!$ is large, i.e., n is large,
- $|f^{(n+1)}(x)|$ is numerically small in the neighborhood of x_0

If these conditions are satisfied, we can write

$$f(x) \approx T_n(x) \tag{16}$$

in the neighborhood of x_0 and the error is small. The formula (16) presents the solution of the problem stated in the motivation for this section — it is the best polynomial approximation of the function f in a neighborhood of the point x_0.

Extremal problems

Definition (local extrema)

Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathrm{Dom}(f)$. The function f is said to take on its *local maximum* at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) \geq f(x)$ for all $x \in N(x_0)$.
The function f is said to take on its *sharp local maximum* at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) > f(x)$ for $x \in N(x_0) \setminus \{x_0\}$.
If the opposite inequalities hold, then the function f is said to take on its *local minimum* or *sharp local minimum* at the point x_0.
A common word for local minimum and maximum is a *local extremum* (pl. *extrema*). A common word for the sharp local maximum and the sharp local minimum is a *sharp local extremum*.

Theorem (sufficient conditions for (non-)existence of local extrema)

Let f be a function defined and continuous in some neighborhood of x_0.

- *If the function f is increasing in some left-hand side neighborhood of the point x_0 and decreasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local maximum at the point x_0.*
- *If the function f is decreasing in some left-hand side neighborhood of the point x_0 and increasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local minimum at the point x_0.*
- *If the function f is either increasing or decreasing in some (two-sided) neighborhood of the point x_0, then there is no local extremum of the function f at x_0.*

Definition (stationary point)

The point x_0 is said to be a *stationary point* of the function f if $f'(x_0) = 0$.

Theorem (relationship between stationary point and local extremum)

Let f be a function defined in x_0. If the function f takes on a local extremum at $x = x_0$, then the derivative of the function f at the point x_0 either does not exist or equals zero and hence $x = x_0$ is a stationary point of the function f.

Theorem (relationship between derivative and monotonicity)

Let f be a function. Suppose that f is differentiable on the open interval I.

- *If $f'(x) > 0$ on I, then the function f is increasing on I.*
- *If $f'(x) < 0$ on I, then the function f is decreasing on I.*

Concavity

Another property of functions which possesses a clear interpretation on the graph is concavity.

Definition (concavity)

Let f be a function differentiable at x_0.
The function f is said to be *concave up* (*concave down*) at x_0 if there exists a ring neighborhood $\overline{N}(x_0)$ of the point x_0 such that for all $x \in \overline{N}(x_0)$ the points on the graph of the function f are above (below) the tangent to the graph in the point x_0, i.e. if

$$f(x) > f(x_0) + f'(x_0)(x - x_0) \qquad \Big(f(x) < f(x_0) + f'(x_0)(x - x_0)\Big) \tag{17}$$

holds.
The function is said to be *concave up (concave down) on the interval* I if it has this property in each point of the interval I.

Definition (point of inflection)

The point x_0 in which the type of concavity changes is said to be a *point of inflection* of the function f.

Theorem (relationship between the 2nd derivative and concavity)

Let f be a function and f'' be the second derivative of the function f on the open interval I.

- *If $f''(x) > 0$ on I, then the function f is concave up on I.*
- *If $f''(x) < 0$ on I, then the function f is concave down on I.*

Theorem (2nd derivative test, concavity and local extrema)

Let f be a function and x_0 a stationary point of this function.

- *If $f''(x_0) > 0$, then the function f has its local minimum at the point x_0.*
- *If $f''(x_0) < 0$, then the function f has its local maximum at the point x_0.*
- *If $f''(x_0) = 0$, then the 2nd derivative test fails. A local extremum may or may not occur. Both cases are possible.*

Behavior of the function near infinity, asymptotes

In the remaining part of this chapter we will investigate functions near the points $+\infty$ and $-\infty$. We will be interested in the fact, whether the graph approaches a line or not.

Definition (inclined asymptote)

Let f be a function defined in some neighborhood of $+\infty$. The line $y = kx + q$ is said to be an *inclined asymptote at $+\infty$ to the graph of the function $y = f(x)$* if

$$\lim_{x\to\infty} |kx + q - f(x)| = 0$$

holds.
Similarly, if we consider the point $-\infty$ instead of $+\infty$, we obtain the definition of the inclined asymptote at $-\infty$.

Remark

As a special case of the preceding definition we obtain for $k = 0$ the horizontal asymptote, introduced on the page 25.

Theorem (inclined asymptote)

Let f be a function defined in some neighborhood of the point $+\infty$. The line $y = kx + q$ is an inclined asymptote at $+\infty$ to the graph of the function $f(x)$ if and only if the following limits exist as finite numbers

$$k := \lim_{x\to\infty} \frac{f(x)}{x} \qquad \textit{and} \qquad q := \lim_{x\to\infty} (f(x) - kx). \tag{18}$$

Similarly, if we consider $-\infty$ instead of $+\infty$, we obtain the asymptote at $-\infty$.

Investigation of a function, curve sketching

Now, we will be interested in the problem to find out the most important properties and some detailed information about the function. Our main object is not to transfer a complete tabulation of functional values to graph paper but to gain a good qualitative (and maybe also quantitative) understanding of the function shape with as little effort as possible. To gain a rapid impression of a function and to explore its main properties we employ various methods of the differential calculus.

Usually we investigate a function f in the following steps.

1. We find the domain of f, decide whether f is odd, even or periodical.
2. We find intercepts of the graph with axes and intervals where the value of the function is positive and/or negative.
3. We find the one-sided limits at the points of discontinuity and at $\pm\infty$.
4. We find and simplify the derivative f'. Then we find intervals where f is increasing and/or decreasing and local extrema of the function f.
5. We find and simplify the second derivative f''. Then we find the intervals where f is concave up and/or down and inflection points of the function f.
6. If the limits in $+\infty$ and/or $-\infty$ are not finite, we find inclined asymptotes in these points.
7. We sketch the graph of the function f.

When constructing the graph of given function we are interested less in drawings of high accuracy than in drawings which "show trends". From the graph of the function it must be clear where the function is increasing and where decreasing, where there are breaks in the graph, where there are zeros and local extrema of the function,where it is concave up and down and what is its shape generally. An idea to locate a few points of the graph and connect them blindly is usually not helpful. The fact that it is sometimes better not to keep the same scale for all parts of the axes can be demonstrated on the comparison of the hand-created pictures to the following examples and the graphs generated by computer. (Try to generate the graphs on the computer yourself. M$ Excel is not the best tool for this, but still can be used.)

Using the computer algebra systems

- Wolfram Alpha

  `http://www.wolframalpha.com/`

- Mathematical Assistant on Web

  `http://um.mendelu.cz/maw-html/menu.php?lang=en&form=`

In the following examples we show using Wolfram Alpha.

Example (Domain)

Find the domain of the following functions:

$$y = \ln\left(\frac{x+2}{x-3}\right), \quad y = \sqrt{x^2 - 3x + 2}.$$

Solution:

```
domain of f(x)=ln((x+2)/(x-3))
domain of f(x)=sqrt(x^2-3*x+2)
```

Example (Graph)

Sketch the graph of the functions:

$$y = x^2 + 2, \quad y = \sin x.$$

Solution:

```
plot x^2+2
plot sin(x)
```

Example (Limits)

Find the limits:

$$\lim_{x\to\infty} \frac{x^5 - 3x^4 + x - 2}{2x^3 + 4}, \quad \lim_{x\to 0^+} \frac{\ln x}{x}, \quad \lim_{x\to 1} \frac{x+2}{x-1}.$$

Solution:

```
lim (x^5-3x^4+x-2)/(2x^3+4) as x->infty

lim (ln x)/x as x->0+

lim ((x+2)/(x-1)) as x->1
```