Definite integral

Mathematics – FRDIS

MENDELU

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Motivation - area under a curve

Suppose, for simplicity, that $y = f(x)$ is a **nonnegative** and **continuous** function defined on $[a, b]$.

> **What is the area of the region bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$?**

The area can be approximated with the sum of rectangles:

- We cut the interval $[a, b]$ into subintervals. Each of these subintervals forms the base of a rectangle, where the height of the rectangle is equal to the value of the function f evaluated at an arbitrary point from the given subinterval.
- The approximation improves as the rectangles become narrower and the number of rectangles increases. We define the area of the region to be the limit of the rectangle area sums as the rectangles become smaller and smaller and the number of rectangles we use approaches infinity.

Such a limit can be defined even for more general functions and we call it **definite integral**. The definite integrals can be defined in many different ways, we will define the Riemann definite integral.

Construction of the Riemann integral

Let f be a function defined on an interval $[a, b]$ and suppose that f is **bounded** on this interval.

- The sequence of points $D = \{x_0, x_1, x_2, \dots, x_n\}$ such that

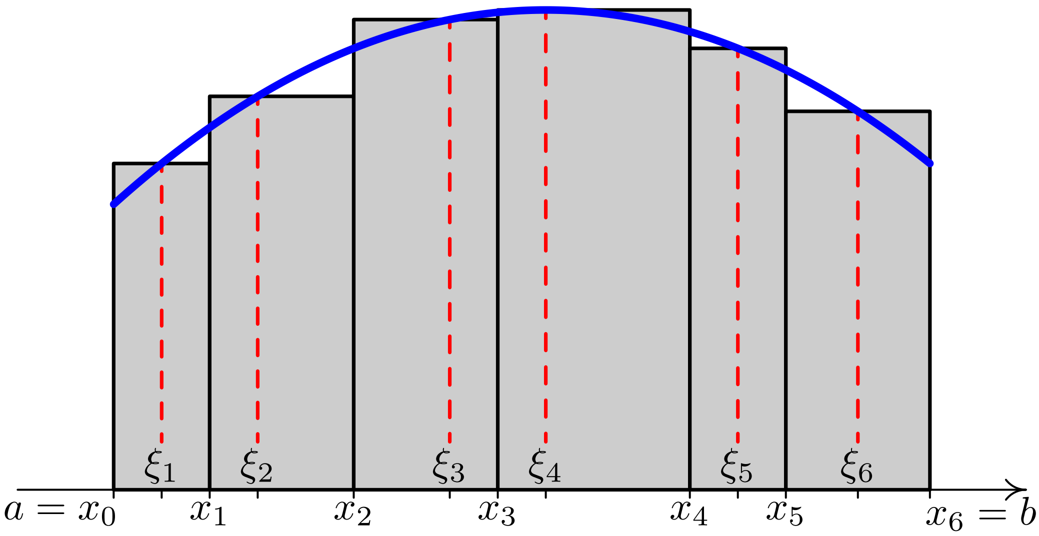
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

  is said to be a **partition of the interval** $[a, b]$. The intervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ are called **subintervals of the partition**.
- The number $\nu(D) = \max\{x_i - x_{i-1}, i = 1, 2, \dots, n\}$ is called a **norm of the partition** D, i.e., the norm of the partition is the length of the longest subinterval of the partition.
- We choose an arbitrary number from each of the subintervals $\xi_1 \in [x_0, x_1], \xi_2 \in [x_1, x_2], \dots, \xi_n \in [x_{n-1}, x_n]$ and we denote $\Xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ the set of these numbers. Then the sum

$$\sigma(f, D, \Xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

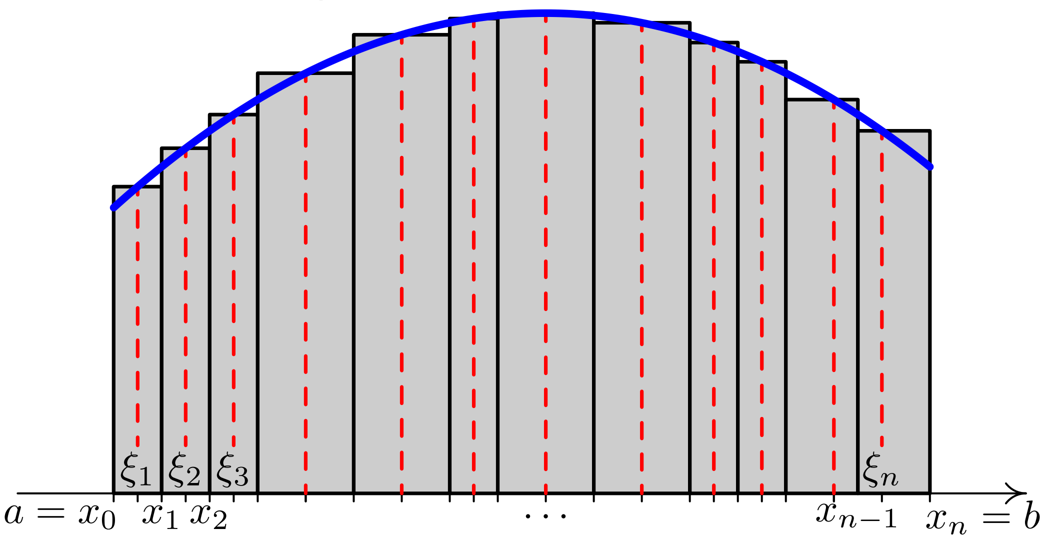
  is called the **integral sum** associated to the function f, the partition D and the choice of the numbers ξ_i in Ξ.

Integral sum:



$$\begin{aligned} \sigma(f, D, \Xi) &= f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + f(\xi_3)(x_3 - x_2) \\ &\quad + f(\xi_4)(x_4 - x_3) + f(\xi_5)(x_5 - x_4) + f(\xi_6)(x_6 - x_5) \\ &= \sum_{i=1}^{6} f(\xi_i)(x_i - x_{i-1}) \end{aligned}$$

Refinement of the partition:



$$\begin{aligned} \sigma(f, D, \Xi) &= f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1}) \\ &= \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \end{aligned}$$

Definition (Riemann integral)

Let f be a function defined and bounded on an interval $[a, b]$. Let

- $D_1, D_2, D_3, \ldots, D_n, \ldots$ be a sequence of partitions of $[a, b]$ which satisfies $\lim_{n\to\infty} \nu(D_n) = 0$ and
- $\Xi_1, \Xi_2, \Xi_3, \ldots, \Xi_n, \ldots$ be a sequence of the corresponding choices of numbers ξ_i from subintervals of the partitions.

The function f is said to be **integrable on** $[a, b]$ **(in sense of Riemann)** if there exists a number $I \in \mathbb{R}$ with the property

$$\lim_{n\to\infty} \sigma(f, D_n, \Xi_n) = I$$

for every sequence of partitions (with the above given property) and for arbitrary particular choice of the points ξ_i in Ξ_n. The number I is said to be a **Riemann integral** of the function f on $[a, b]$ and it is denoted

$$I = \int_a^b f(x)\,\mathrm{d}x.$$

The number a is called a **lower limit** of the integral and the number b is called an **upper limit** of the integral.

We have to distinguish the definite integrals from the indefinite integrals:

- **Indefinite integral is a set of functions.**
- **Definite integral is a limit (number).**

We will see, that there is a connection between the definite and indefinite integrals, since definite integrals can be evaluated using indefinite integrals.

Theorem (Sufficient conditions for integrability)

Let f be a function which satisfies at least one of the following conditions:

1. f *is continuous on* $[a,b]$,
2. f *is monotone on* $[a,b]$,
3. f *is bounded on* $[a,b]$ *and contains at most a finite number of discontinuities on this interval.*

Then the function f is integrable (in sense of Riemann) on $[a,b]$, i.e., $\int_a^b f(x)\,\mathrm{d}x$ *exists.*

Properties of the Riemann integral

Theorem (Additivity and homogenity with respect to the integrand)

Let f and g be functions which are integrable on $[a,b]$, $c \in \mathbb{R}$. Then the functions $f+g$ and cf are also integrable on $[a,b]$ and it holds:

$$\begin{aligned} \int_a^b [f(x)+g(x)]\;\mathrm{d}x &= \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x \\ \int_a^b cf(x)\,\mathrm{d}x &= c\int_a^b f(x)\,\mathrm{d}x \end{aligned}$$

Theorem (Additivity with respect to the domain of integration)

Let f be a function defined of $[a,b]$, and let $c \in (a,b)$ be any number. Then the function f is integrable on $[a,b]$ if and only if it is integrable on both the intervals $[a,c]$ and $[c,b]$ and it holds:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

Theorem

Let f and g be functions integrable on $[a, b]$ such that $f(x) \leq g(x)$ on this interval. Then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

It follows from the last theorem that

$$\text{if} \quad g(x) \geq 0, \quad \text{then} \quad \int_a^b g(x)\,\mathrm{d}x \geq 0,$$

i.e., integral of the nonnegative function is nonnegative.

Evaluation of the Riemann integral

Theorem (Newton - Leibniz formula)

Let f be a function integrable on $[a, b]$ and let F be an antiderivative of f on (a, b) which is continuous on $[a, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a).$$

The previous theorem says that to calculate the Riemann integral of f over $[a, b]$, all we need to do is:

1. find an antiderivative F of f,
2. calculate the number $F(b) - F(a)$.

Example

1. $\displaystyle\int_2^3 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_2^3 = \frac{3^3}{3} - \frac{2^3}{3} = \frac{19}{3}.$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = [-\cos x]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 1.$

3.
$$\begin{aligned}\int_{-3}^{1} |x|\,\mathrm{d}x = \int_{-3}^{0}(-x)\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = \left[-\frac{x^2}{2}\right]_{-3}^{0} + \left[\frac{x^2}{2}\right]_0^1 \\ = 0 + \frac{9}{2} + \frac{1}{2} - 0 = 5.\end{aligned}$$

Theorem (Integration by parts)

Let u, v be functions having continuous derivatives on $[a, b]$. Then

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

Example

$$\begin{aligned}\int_1^2 x\ln x\,\mathrm{d}x &= \left|\begin{array}{ll} u = \ln x & v' = x \\ u' = \frac{1}{x} & v = \frac{x^2}{2}\end{array}\right| = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\frac{x^2}{2}\,\mathrm{d}x \\ &= \frac{4}{2}\ln 2 - \frac{1}{2}\ln 1 - \frac{1}{2}\int_1^2 x\,\mathrm{d}x = 2\ln 2 - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 \\ &= 2\ln 2 - \frac{1}{2}\left[\frac{4}{2} - \frac{1}{2}\right] = 2\ln 2 - \frac{3}{4}.\end{aligned}$$

Theorem (Substitution method)

Let f be a function continuous on $[a, b]$ and let φ be a function which has continuous derivative φ' on $[\alpha, \beta]$. Further suppose that $a \leq \varphi(x) \leq b$ for $x \in [\alpha, \beta]$. Then

$$\int_{\alpha}^{\beta} f[\varphi(x)]\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

- The formula in the theorem can be used "from left to right" (the 1 st substitution method) and "from right to left"(the 2 nd substitution method).
- It may happen in some particular cases that the lower limit $\geq$ the upper limit after the substitution. For this reason we introduce the following extension:

Extension

The symbol $\int\limits_a^b f(x)\,\mathrm{d}x$ can be extended for $b \leq a$ as follows:

$$\int_a^a f(x)\,\mathrm{d}x = 0, \qquad \int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

We have two possibilities when evaluating the definite integral with using the substitution method:

1. We use the previous theorem, i.e., we transform the limits of the integral and then we use the Newton-Leibniz formula with the new limits. (We do not substitute the original variable into the antiderivative obtained after the integration.)

2. We do not use the previous theorem. We evaluate the indefinite integral (i.e., we substitute the original variable after integration) and then we apply the Newton-Leibniz formula with the original limits.

Example

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos x \, dx$.

1. We transform the limits:

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos x \, dx = \left| \begin{array}{c} t = \sin x \\ dt = \cos x \, dx \\ t_1 = \sin 0 = 0 \\ t_2 = \sin \frac{\pi}{2} = 1 \end{array} \right| = \int_0^1 t^2 \, dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}.$$

2. We do not transform the limits:

$$\int \sin^2 x \cdot \cos x \, dx = \left| \begin{array}{c} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int t^2 \, dt = \frac{t^3}{3} = \frac{\sin^3 x}{3} + c.$$

Tedy

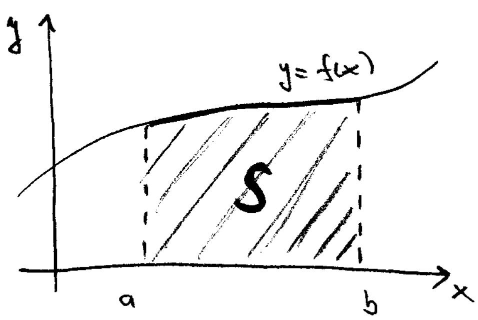
$$\int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos x \, dx = \left[\frac{\sin^3 x}{3} \right]_0^{\frac{\pi}{2}} = \frac{\sin^3 \frac{\pi}{2}}{3} - \frac{\sin^3 0}{3} = \frac{1}{3}.$$

Applications of the Riemann integral in geometry

The area under a curve and between two curves

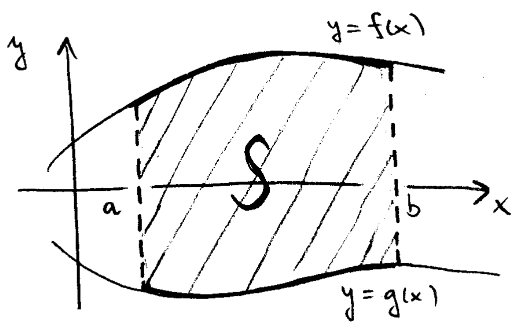
- Let f be a nonnegative and continuous function on $[a, b]$.
  The area S of the region in the plane bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is:

$$S = \int_a^b f(x) \, dx$$



- Let f, g be continuous functions and suppose $f(x) \geq g(x)$ for $x \in [a, b]$.
  The area S of the region in the plane bounded by $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is:

$$S = \int_a^b [f(x) - g(x)] \, dx$$

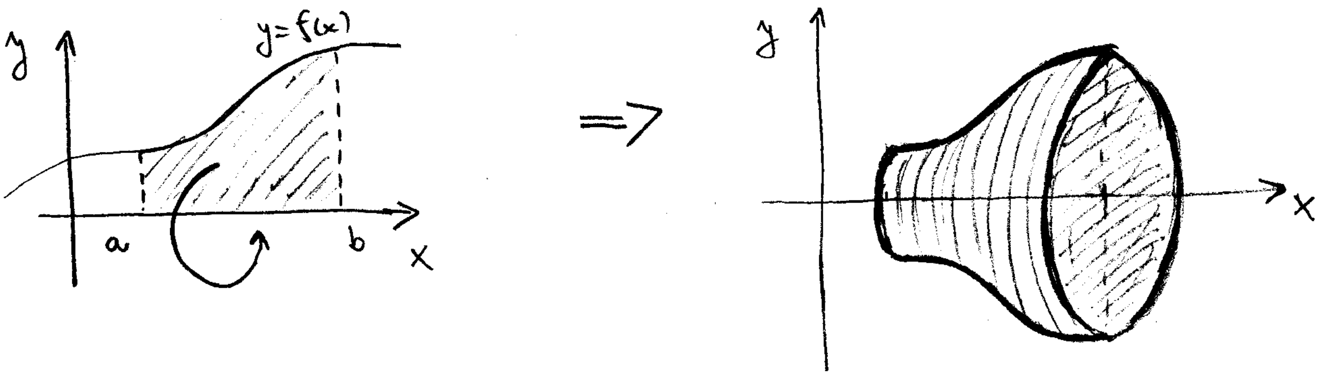


(The signs of f and g are arbitrary.)

Volume of the solid of revolution I

Let f be a nonnegative and continuous function on $[a, b]$.
The volume V of the solid generated by revolving the region bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ about the x-axis is:

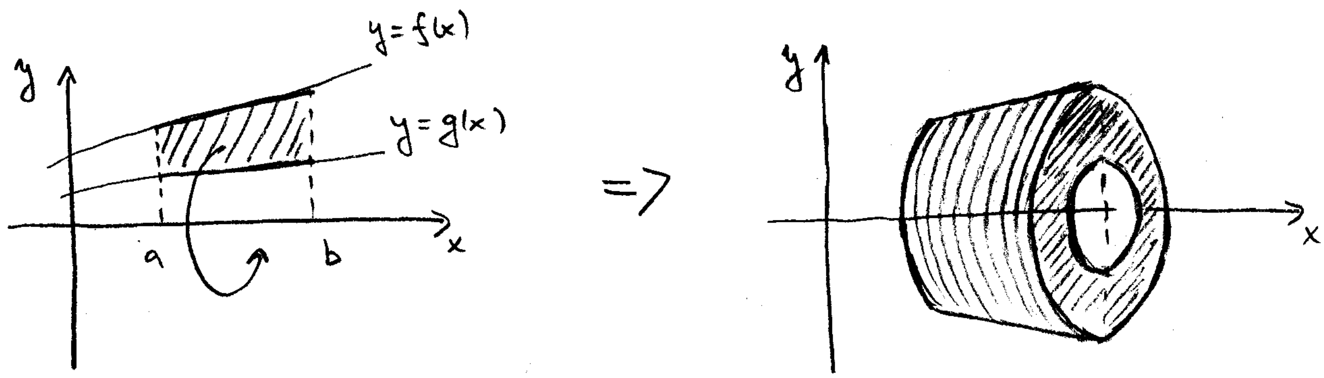
$$\boxed{V = \pi \int_a^b f^2(x)\, \mathrm{d}x}$$



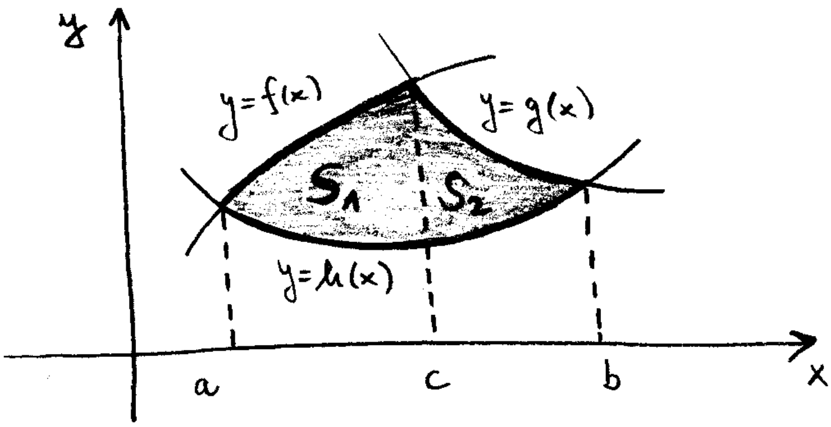
Volume of the solid of revolution II

Let f, g be nonnegative continuous functions and suppose $f(x) \geq g(x)$ for $x \in [a, b]$.
The volume V of the solid generated by revolving the region bounded by $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ about the x-axis is:

$$\boxed{V = \pi \int_a^b \left[f^2(x) - g^2(x)\right]\, \mathrm{d}x}$$



Many other areas and volumes can be calculated using the above formulas since we can cut the given region into several pieces which satisfy the above conditions.

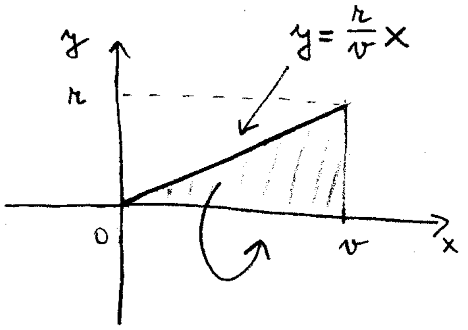


$$S = S_1 + S_2 = \int_a^c [f(x) - h(x)]\,\mathrm{d}x + \int_c^b [g(x) - h(x)]\,\mathrm{d}x$$

Example (Volume of the cone)

Find the formula for a volume of a cone such that the radius of the base is r and the altitude of the cone is v.

Solution:
If the following triangle revolves about the x-axis, we obtain the cone:



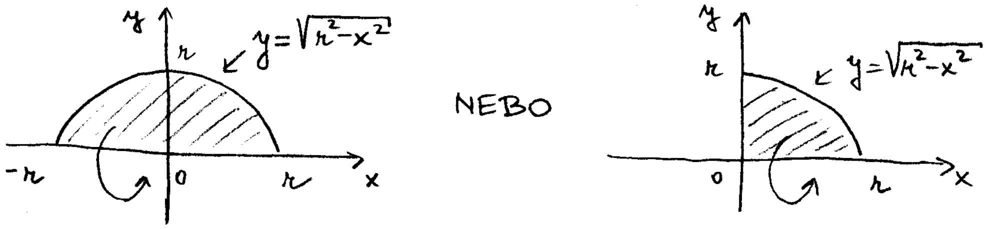
$$V = \pi \int_0^v \left(\frac{r}{v}x\right)^2 \mathrm{d}x = \frac{\pi r^2}{v^2}\int_0^v x^2\,\mathrm{d}x = \frac{\pi r^2}{v^2}\left[\frac{x^3}{3}\right]_0^v = \frac{\pi r^2}{v^2}\frac{v^3}{3} = \frac{\pi r^2 v}{3}$$

Example (Volume of the ball)

Find the formula for the volume of a ball with the radius r.

Solution: The equation of a circle with radius r and the center at $[0, 0]$ is $x^2 + y^2 = r^2$. The upper half-circle is the graph of the function $y = \sqrt{r^2 - x^2}$, the lower half-circle is the graph of the function $y = -\sqrt{r^2 - x^2}$.

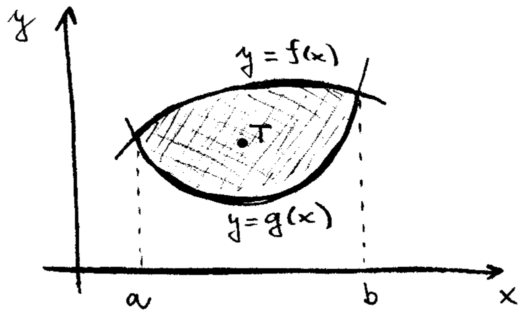
If the half-circle revolves about the x-axis, we obtain a ball. If a quarter of a circle revolves about the x-axis, we obtain a half of the ball.



$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,\mathrm{d}x = 2\pi \int_{0}^{r} (r^2 - x^2)\,\mathrm{d}x$$
$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4\pi r^3}{3}$$

Application of the definite integral in physics

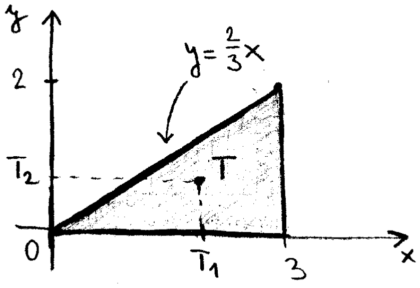
Consider a region bounded by the graphs of the functions $y = f(x), y = g(x)$, and the lines $x = a$, $x = b$, where $g(x) \leq f(x)$ on $\langle a, b \rangle$.



Suppose that the region has a constant density ρ. Then:

- Mass of the region:
$$m = \rho \int_a^b [f(x) - g(x)]\ \mathrm{d}x$$
- Moments of force with respect to x-axis, y-axis:
$$S_x = \frac{1}{2}\rho \int_a^b \left[f^2(x) - g^2(x)\right]\ \mathrm{d}x, \quad S_y = \rho \int_a^b x\,[f(x) - g(x)]\ \mathrm{d}x$$
- Center of mass:
$$T = \left[\frac{S_y}{m}, \frac{S_x}{m} \right]$$

Example (Center of mass)



Find the center of mass of the triangle given by the vertices $[0,0],[3,0],[3,2]$. Suppose that the density ρ is constant.

- Mass:
$$m=\rho\int_0^3 \frac{2}{3}x\,\mathrm{d}x=\rho\frac{2}{3}\left[\frac{x^2}{2}\right]_0^3=\rho\frac{2}{3}\cdot\frac{9}{2}=3\rho$$
- Moments of force:
$$S_x=\frac{1}{2}\rho\int_0^3 \frac{4}{9}x^2\,\mathrm{d}x=\frac{2}{9}\rho\left[\frac{x^3}{3}\right]_0^3=\frac{2}{9}\rho\cdot 9=2\rho$$
$$S_y=\rho\int_0^3 x\cdot\frac{2}{3}x\,\mathrm{d}x=\frac{2}{3}\rho\int_0^3 x^2\,\mathrm{d}x=\frac{2}{3}\rho\left[\frac{x^3}{3}\right]_0^3=\frac{2}{3}\rho\cdot 9=6\rho$$
- Center of mass:
$$T_1=\frac{S_y}{m}=2,\quad T_2=\frac{S_x}{m}=\frac{2}{3}\Longrightarrow T=\left[2,\frac{2}{3}\right]$$

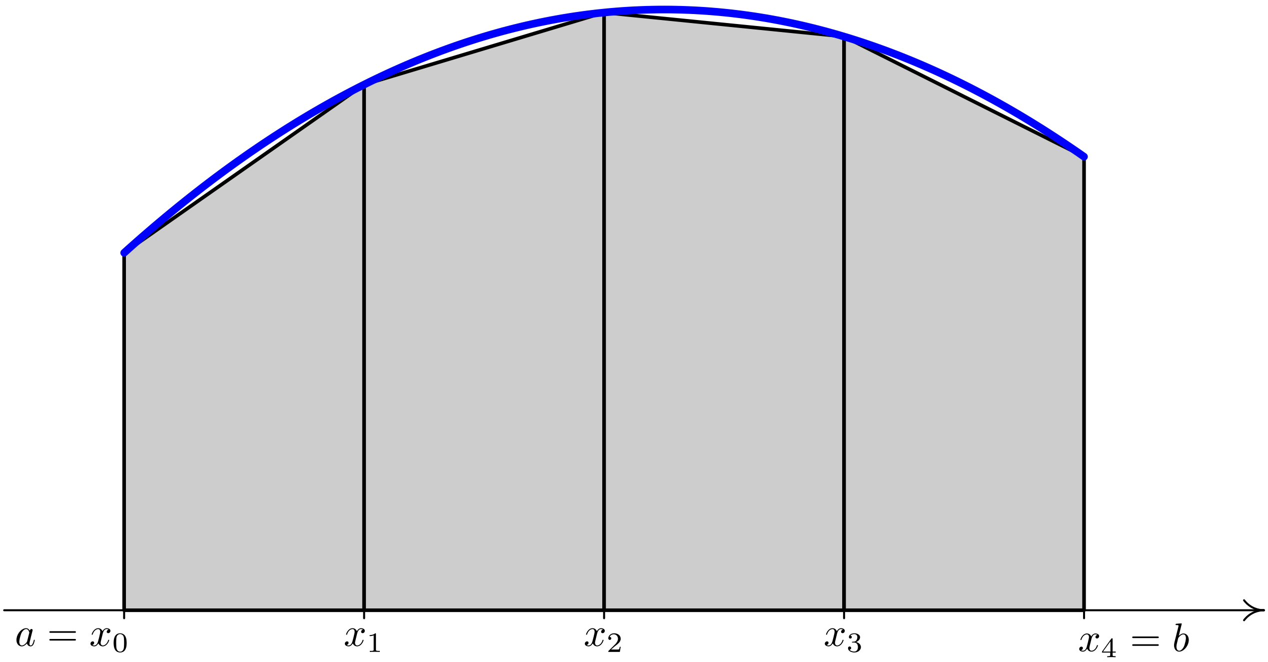
Approximation of definite integrals

Numerical methods for approximating the definite integral $\int\limits_a^b f(x)\,\mathrm{d}x$ are used in the following cases:

- The antiderivative of the function f has no elementary formula, hence the Newton-Leibniz formula cannot be used ($\frac{\sin x}{x}$, $\sin x^2$, $\frac{e^x}{x}$, $e^{-x^2}, \dots$).
- A formula for the function f is not known, we have only a set of measured values.

The trapezoidal rule

The trapezoidal rule for the evaluation a definite integral is based on approximating the region between a curve and the x-axis with trapezoids.



Let f be a function bounded on $[a, b]$. To evaluate $\int_a^b f(x)\,\mathrm{d}x$:

- We cut the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$, $(x_0 = a, x_n = b)$.
- Suppose that the length of each subinterval is $h = \frac{b-a}{n}$.
- Denote $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$.
- We approximate the function f on $[x_{i-1}, x_i], (i = 1, 2, \dots, n)$ with the linear function passing through $[x_{i-1}, y_{i-1}]$, $[x_i, y_i]$. This linear function is of the form

  $$\boxed{y = y_{i-1} + \frac{y_i - y_{i-1}}{h}(x - x_{i-1})}.$$

  Hence

  $$\int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x \approx \int_{x_{i-1}}^{x_i} \left[y_{i-1} + \frac{y_i - y_{i-1}}{h}(x - x_{i-1})\right]\,\mathrm{d}x$$

$$
\begin{aligned}
\int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x &\approx \int_{x_{i-1}}^{x_i} \left[y_{i-1} + \frac{y_i - y_{i-1}}{h}(x - x_{i-1}) \right] \mathrm{d}x \\
&= \left[y_{i-1}x + \frac{y_i - y_{i-1}}{2h}(x - x_{i-1})^2 \right]_{x_{i-1}}^{x_i} \\
&= y_{i-1}h + \frac{y_i - y_{i-1}}{2h}h^2 \\
&= \frac{h}{2}(y_{i-1} + y_i),
\end{aligned}
$$

which (in case when f is a positive function) is the well-known formula for evaluating an area of the trapezoid with corners $[x_{i-1}, 0], [x_{i-1}, y_{i-1}], [x_i, 0], [x_i, y_i]$. Hence,

$$
\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &= \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x \\
&\approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n) \\
&= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).
\end{aligned}
$$

The trapezoidal rule

Let f be a function bounded on $[a, b]$, and let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$ such that the length of each subinterval of this partition is $h = \frac{b-a}{n}$. Then

$$
\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n),
$$

where $y_0 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$. (We suppose that f is defined at the points $x_0, x_1, \ldots, x_n$.)

- Some other rules for approximating the definite integrals can be used, e.g., the so-called Simpson's rule is based on approximating curves with parabolas instead of lines.

Using the computer algebra systems

Wolfram Alpha:

`http://www.wolframalpha.com/`

Mathematical Assistant on Web (MAW):

`wood.mendelu.cz/math/maw-html/index.php?lang=en&form=main`

Example

Using the Wolfram Alpha find the integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x.$$

Solution:

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integrate sin x dx from x=0 to pi
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