Exercises – linear algebra

Mathematics

FRDIS

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MENDELU

1 Vectors, matrices, determinants

1.1 Operations with matrices

1. Let
$$A = \begin{pmatrix} 3 & 1 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 5 & 3 \\ 4 & -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$
Decide which of the following products can be calculated and find the size of the resulting matrices:
$$AB,\ BA,\ AC,\ CA,\ AD,\ DA,\ BC,\ CB,\ BD,\ DB,\ CD,\ DC,\ C^T D,\ B^T D, B^T A.$$

2. Let
$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 2 & 3 \end{pmatrix}.$$
Calculate $(A - 2I)^T \cdot B$, where I is the identity matrix .

3. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}.$$
Calculate $(A^T + I) \cdot B$, where I is the identity matrix.

4. Let
$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$
Calculate $(A - B)^2$, where I is the identity matrix.

5. Let
$$A = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$
Calculate A^2.

6. Let
$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$
Calculate $(A^T - I)A$, where I is the identity matrix.

1.2 Determinant, inverse matrix, linear dependence/independence of vectors

1. Let
$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

2. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

3. Let
$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

4. Let
$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

5. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 5 \end{pmatrix}.$$

(a) Evaluate the determinant of A.

(b) Using the value of $\det A$ answer the following questions:

   (i) Are the rows of A linearly dependent or independent?

   (ii) Is rank $(A) > 3$, rank $(A) < 3$ or rank $(A) = 3$?

   (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

6. Are the following vectors liearly dependent or independent?

   (a) $\vec{a} = (1, 2, 1, 0)$, $\vec{b} = (1, 2, -1, 1)$, $\vec{c} = (0, 1, 2, 1)$, $\vec{d} = (1, 1, 0, 1)$

   (b) $\vec{a} = (1, 2, 1, 0)$, $\vec{b} = (1, 0, -1, 1)$, $\vec{c} = (1, 1, 2, 1)$, $\vec{d} = (2, 1, 1, 2)$

   (c) $\vec{a} = (1, 3, 1, 0)$, $\vec{b} = (1, -1, 0, 1)$, $\vec{c} = (1, 1, 2, 1)$, $\vec{d} = (1, 1, 1, 2)$

2 Systems of linear equations

Solve the following systems using the Gauss method.

(a) Find the rank of the coeffcient and of the augmented matrix and determine how many solutions the system has.

(b) Find the solution of the system (if exists any).

1.

$$\begin{aligned} 8x_1+6x_2-x_3+3x_4&=-9\\ 2x_1+2x_2-x_3+5x_4&=-13\\ x_1+2x_2-2x_3+11x_4&=-28\\ 2x_2-3x_3+17x_4&=-43. \end{aligned}$$

2.

$$\begin{aligned} x_1+x_2-x_3+x_4&=-2\\ 2x_1+x_2-x_3+2x_4&=2\\ 3x_1+2x_2-2x_3+3x_4&=1\\ x_2-3x_3+2x_4&=-3. \end{aligned}$$

3.

$$\begin{aligned} x_1+2x_2-x_4&=-2\\ 2x_1+3x_2+x_3-5x_4&=1\\ x_1+x_2+x_3-4x_4&=3\\ x_2-x_3+2x_4&=0. \end{aligned}$$

4.

$$\begin{aligned} x_1+x_2-2x_3+3x_4&=0\\ 3x_1+2x_2+3x_3-4x_4&=-4\\ -3x_1-2x_2-3x_3+3x_4&=4\\ -7x_1-6x_2+5x_3-8x_4&=4. \end{aligned}$$

5.

$$\begin{aligned} x_1+x_2+3x_3-x_4&=2\\ 2x_1+x_2+5x_3-2x_4&=0\\ 2x_1-x_2+3x_3-2x_4&=-8\\ 3x_1+2x_2+8x_3-3x_4&=2. \end{aligned}$$

6.

$$\begin{aligned} x_1+3x_2-2x_3+x_4&=0\\ 2x_1+5x_2-3x_3+3x_4&=0\\ x_1+2x_3-2x_4&=9\\ 2x_1-x_2+4x_3+9x_4&=3. \end{aligned}$$

7.

$$\begin{aligned} x_1+3x_2+2x_3-4x_4&=-4\\ x_2+x_3-3x_4&=-3\\ -x_1+2x_2+x_3-x_4&=-1\\ 5x_1+2x_2+4x_4&=4. \end{aligned}$$

8.

$$\begin{aligned} x_1+2x_2-5x_3+x_4&=-2\\ x_2+3x_3-4x_4&=1\\ -x_1+2x_2-x_3+x_4&=6\\ 3x_1+x_2-4x_3+6x_4&=-2. \end{aligned}$$

9.

$$\begin{aligned} x_1+x_2-x_3+x_4&=0\\ 2x_1+3x_2+x_3+x_4&=6\\ 4x_1+5x_2-x_3+3x_4&=6\\ 3x_1+4x_2-6x_3+2x_4&=-6. \end{aligned}$$

10.

$$\begin{aligned} x_1-x_2+x_3+2x_4&=1\\ x_1-2x_2-x_3+2x_4&=1\\ 2x_1+3x_3+x_4&=2\\ x_1+x_2+3x_3&=1. \end{aligned}$$

11.

$$\begin{aligned} x_1+x_2+2x_4&=0\\ x_1+x_3+x_4&=2\\ 2x_1+x_2+x_3+3x_4&=3\\ x_2-2x_3+3x_4&=1. \end{aligned}$$

12.

$$\begin{aligned} x_1+x_2+5x_4&=1\\ x_1+x_3+2x_4&=1\\ x_1-3x_2+4x_3-7x_4&=1\\ x_2-x_3+3x_4&=0. \end{aligned}$$